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Geometric Interpretation of Adjoint Equations in Optimal Low Thrust Space Flight

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Time-optimal control of two seemingly unrelated problems are solved using Pontryagin's Maximum Principle. The first is a simple double integrator in $\mathbf{R}^2$ in which the state is driven to a desired terminal state in minimum time. The second is an orbiting spacecraft in $\mathbf{R}^2$ which transitions from its current orbit into a desired terminal orbit in minimum time. In both cases, thrust is continuously available but limited in magnitude. The two problems are related by the gravitational parameter of the major body orbited. As the gravitational parameter is mathematically varied to zero, the orbiting spacecraft takes on the dynamics of a double integrator.

A two-point boundary value problem is created when Pontryagin's Maximum Principle is applied to solve the two problems. Shooting methods are typically used in the solution, but they require reasonably close *a priori* estimates of the initial or final values of the costate for the shooting method to converge. The adjoint equations of the double integrator have a simple solution. The derived optimal control is shown to be related to the adjoint solution in a simple geometric manner. A method is presented to estimate the initial costate and terminal time for the double integrator problem. The possibility that the initial estimate for the double integrator may provide an initial estimate for the related orbital transfer problem is explored. Numerical examples of the two problems illustrate the method.

I. Introduction

Variational methods are useful tools for the minimization of the time of flight of low thrust space vehicles. Pontryagin's Maximum Principle is one of these tools, and its benefits are threefold. First, it reduces the underlying infinite-dimensional function space problem to one of finite dimensions. It specifies the functional form of the solution in terms of a set of differential equations, leaving a few parameters to be found. Second, it places necessary conditions on these remaining parameters, providing important clues to building an algorithm to determine their values. Third, it is a variational tool hence locates local minima, yet frequently the method finds the desired global minimum despite this.

Despite the power of the method, there remains the formidable problem of determining the parameters. These parameters typically include the time of flight and some combination of initial and final conditions of the resulting set of differential equations. These equations include the system differential equations (n states) and the adjoint differential equations (n costates), resulting in a combined system of $2n$ states. Hardly ever are the $2n$ parameters all initial or all final conditions, hence a two-point boundary value problem must be solved. Efficient computational methods exist for solving such problems, but most depend on having a good initial guess of the parameters.

The differential equations representing the equations of motion have a clear interpretation: they encode the physics of the problem. The adjoint equations, however, are more difficult to interpret: they encode the optimality of the problem. One may view the adjoint equations as generalized Lagrange multipliers, with the obvious interpretation of the adjoint equations representing the dual optimization problem. We seek a more geometric interpretation in this paper, one that geometrically links the optimality of the problem with

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the physics. Such interpretation naturally leads to a better understanding of the problem so that one may more easily visualize the optimal solution.

In this paper, Pontryagin's Maximum Principle is applied to two problems: the double integrator in $\mathbf{R}^2$ and an orbit transfer problem also in $\mathbf{R}^2$. In each, we seek the control that drives the state from its initial value to some desired terminal value in minimum time. Also in each, the control is assumed to have a uniform limit in magnitude. The two problems do not generally have a switched solution, as with other time-optimal control problems, thus their solutions tend to be more analytically complicated.

The double integrator problem is easily solved if it is in $\mathbf{R}^1$. The solution is a switched control (bang-bang control). However, in $\mathbf{R}^2$ it is a much tougher problem to solve (and even more complicated in higher dimensions). The solution is not bang-bang except for the special case when the problem degenerates into the $\mathbf{R}^1$ problem. The problem does not appear to have an analytic solution, so it is necessary to resort to numerical methods. It becomes a two-point boundary value problem, of which several methods have been developed for solving. There does not appear, however, to be physical intuition achieved from these existing methods. Furthermore, these numerical methods depend on having a good estimate of the optimization parameters, without which they would not converge. We were able to develop an algorithm that produces a nearly optimal solution, and more importantly, gives a sufficient estimate of the optimization parameters needed for numerical shooting methods to converge to the optimal solution.

There is wide applicability to other optimal control problems related to astrodynamics, such as attitude control and navigation. Geometric inferences can significantly aid in the interpretation of the adjoint equations. Such interpretations can help improve computational approaches and greatly enhance the understanding and visualization of such problems.

Other researchers have recognized the practical difficulty of selecting the parameters which start the solution to the two-point boundary value problem. For time-optimal control, Lastman[1] and later Fotouhi-C. and Szyszkowski[2] describe intelligent algorithms to select the initial parameters, but they focus on problems that result in switched (bang-bang) control. Other authors address interesting modifications to the time-optimal control for orbiting spacecraft, such as the double integrator in $\mathbf{R}^1$ with a modified performance criterion to minimize a combination of time and fuel.[3]

In this paper, we focus on nonsingular time-optimal control in problems where there are enough degrees of freedom in the dynamics that the optimal control is a continuous function of time. The paper is organized as follows. Pontryagin's Maximum Principle is reviewed in Section III and includes a complete statement of the necessary conditions for time-optimal control. This principle is then applied to two problems, both with configuration space in $\mathbf{R}^2$. Time-optimal control of the double integrator is investigated in Section IV followed by that of the orbit transfer problem in Section V.

II. Nomenclature

x	n-dimensional state
λ	n-dimensional costate
u	m-dimensional control
u_{max}	maximum control magnitude
$\mathcal{U}$	admissible control class
x_0	initial state
x_f	terminal state
t_f	terminal time
$\mathcal{X}_f$	target state manifold ($= \{x_f\}$ when target state is explicit)
H	Hamiltonian
$\|g\|$	search index
μ	Earth's gravitational constant ($3.986004418 \times 10^{14}$ m^3/s^2)
r_e	Earth's equatorial radius (6,378,137 m)
h	altitude of circular orbit, m
r	orbital radius, m ($r = r_e + h$)

III. Pontryagin's Maximum Principle

In this section, Pontryagin's Maximum Principle is reviewed in the context of the class of problems we seek to solve. Consider the time-invariant dynamical system

$$\dot{x} = f(x, u), \quad x(0) = x_0, \quad t \geq 0 \tag{1}$$

where $x(t) \in \mathbf{R}^n$ and $u(t) \in \mathbf{R}^m$ for all $t \geq 0$. We assume there is limited control authority with admissible control class

$$\mathcal{U} = \{u : \|u(t)\|_2 \leq u_{max} \quad \forall\ t \geq 0\}. \tag{2}$$

Our model for admissibility is that thrust can be pointed in any commanded direction, but its magnitude

$$\|u(t)\|_2 = \sqrt{u_1^2(t) + \cdots + u_m^2(t)} \tag{3}$$

is limited by u_{max}, a value determined by the capability of the propulsion system.

We wish to apply an admissible control u that drives the state of the system from its initial value x_0 to some desired terminal condition in minimum time t_f. The terminal condition may be an explicit target state x_f, or it may be characterized by a smooth manifold $\mathcal{X}_f$ on which the terminal state must lie.

Pontryagin's Maximum Principle[4] provides necessary conditions for determining the time-optimal control for the system of Eq. (1) with limits imposed by Eq. (2). To apply this principle, first define the Hamiltonian H for the general time-optimal control problem:

$$H(x, u, \lambda) = 1 + \lambda^{\mathrm{T}} f(x, u) \tag{4}$$

where $\lambda(t) \in \mathbf{R}^n$ is the costate vector and is defined for all $t \geq 0$. Necessary conditions for t_f to be the minimum terminal time, $\{u^*(t), 0 \leq t \leq t_f\}$ to be the optimal control, $\{x^*(t), 0 \leq t \leq t_f\}$ to be the optimal state trajectory and $\{\lambda^*(t), 0 \leq t \leq t_f\}$ to be the optimal costate trajectory are

1. $H(x^*(t), u^*(t), \lambda^*(t)) = \min\{H(x^*(t), u(t), \lambda^*(t)) : u \in \mathcal{U}\}$ for all $t \in [0, t_f]$;
2. $H(x^*(t), u^*(t), \lambda^*(t)) = 0$ for all $t \in [0, t_f]$;
3. Adjoint equations: $\dot{\lambda}^*(t) = -\frac{\partial H}{\partial x}(x^*(t), u^*(t), \lambda^*(t))$ for $t \in [0, t_f]$;
4. $\lambda^*(t_f)$ is transversal to the terminal state manifold $\mathcal{X}_f$ (this is no condition if target state is explicit);
5. State equations: $\dot{x}^*(t) = f(x^*(t), u^*(t))$ for $t \in [0, t_f]$;
6. $x^*(0) = x_0$ and $x^*(t_f) \in \mathcal{X}_f$ (when the target state is explicit, $\mathcal{X}_f = \{x_f\}$).

This paper focuses on two time-optimal control problems. Both are nonsingular,[4] hence the necessary conditions lead to a well-defined solution, but it may not be unique. There could be other minima resulting in a lesser terminal time, yet also meet all the necessary conditions. The smallest of these is the global minimum. Although time-optimal control is typically distinguished by switched solutions (e.g., bang-bang control), the two problems addressed in this paper almost always possess smooth functions as the optimal control. Only when the initial and final state conditions are just right will they degenerate to a switched control. In this paper, we focus on the broader non-degenerate case, where the optimal control is a continuous function of time.

IV. Double Integrator in $\mathbf{R}^2$

The necessary conditions for time-optimal control listed in the last section are now applied to the double integrator in $\mathbf{R}^2$. The solution of the adjoint equations is found to be a line segment in the plane, and the optimal control is the projection of this line segment onto the unit circle. The problem thus comes down to determining where the line segment begins (or where it ends) and its slope; this knowledge allows complete solution of the problem. We exploit this geometry to find an *a priori* estimate of where the line segment begins and its slope, allowing shooting methods to be intelligently initialized.

A. Problem Definition

Consider the double integrator defined by

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix}, \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \tag{5}$$

where $x_1(t), x_2(t) \in \mathbf{R}^2$ for all $t \geq 0$ and I is the 2×2 identity matrix. The state can be summarized into one vector with $n = 4$ components:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{6}$$

The control $u(t) \in \mathbf{R}^2$ is admissible if $u_1^2(t) + u_2^2(t) \leq u_{max}^2$ for all $t \geq 0$.

We wish to drive the state in minimum time to a desired terminal state ($\mathcal{X}_f = \{x_f\}$). It is not hard to see that the state can be rotated so that the initial and final positions are aligned horizontally and translated so that the final position is shifted to the origin:

$$x_1(0) = \begin{bmatrix} d \\ 0 \end{bmatrix}, \qquad x_2(0) = \begin{bmatrix} v_t \\ v_n \end{bmatrix}, \qquad x_1(t_f) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad x_2(t_f) = \begin{bmatrix} x_{2f_t} \\ x_{2f_n} \end{bmatrix}. \tag{7}$$

Assuming this transformation is made, the objective is now to drive the position to the origin with a desired final velocity in minimum time. The purpose for posing the problem in this manner is to allow for the geometry of the problem to be understood more easily. The problem is reduced to one where there is no offset between the initial and final vertical positions and there is an initial and final velocity in both the horizontal and vertical directions. An example illustrating this transformation is given in the next subsection. The estimation of the optimization parameters is simplified as a result. More importantly, problems with more complexity and usefulness can sometimes be transformed in a similar manner, allowing the results of the double integrator problem to be extended to a wider class of problems.

B. Application of Maximum Principle

We now apply Pontryagin's Maximum Principle. From Eq. (4), the Hamiltonian is

$$H = 1 + \lambda_1^{\mathrm{T}} x_2 + \lambda_2^{\mathrm{T}} u \tag{8}$$

where $\lambda_1(t), \lambda_2(t) \in \mathbf{R}^2$ for all $t \geq 0$.

From necessary condition 1, it is seen that the optimal control is

$$u^*(t) = -\frac{\lambda_2(t)}{\|\lambda_2(t)\|_2} u_{max} \tag{9}$$

The adjoint equations can be derived from necessary condition 3. They are

$$\frac{d}{dt}\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -I & 0 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \tag{10}$$

where I is the 2×2 identity matrix. Solving the adjoint equations, it is seen that λ_2 is a line with slope λ_1:

$$\lambda_1(t) = c_1, \qquad \lambda_2(t) = c_2 - tc_1 \tag{11}$$

where $c_1, c_2 \in \mathbf{R}^2$. The optimal controller is thus the projection of the line given by Eq. (11) onto the circle with center at the origin and radius u_{max} as given in Eq. (9).

We pause to give an example. Figure 1 illustrates the solution of the double integrator in $\mathbf{R}^2$ with $u_{max} = 1$ and for the initial conditions shown. We went ahead and used all necessary conditions to solve for the solution; we will return to the other necessary conditions in the next subsection. It is important to observe the geometry that relates the solution of the adjoint equations to the optimal control: the optimal

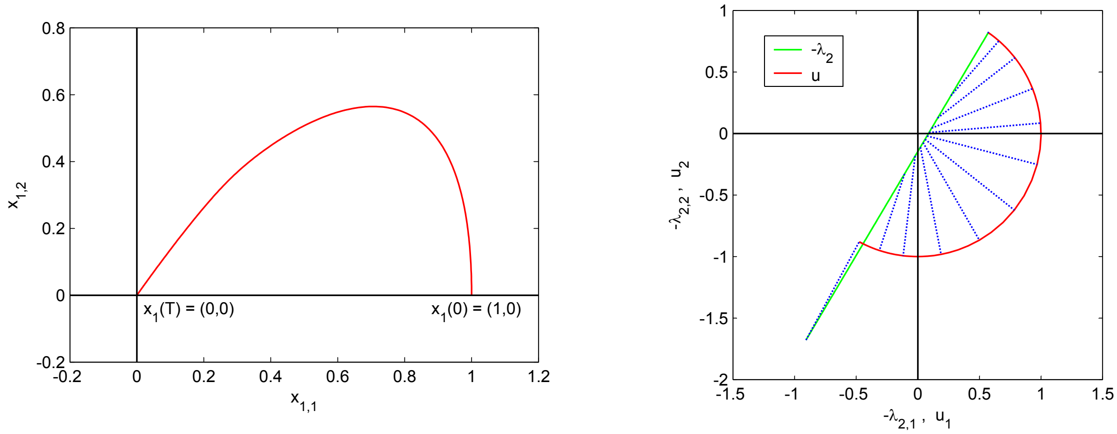


Figure 1. Time-optimal control of the double integrator in $\mathbf{R}^2$. The left plot shows the optimal trajectory of the state as it is driven from initial value $x(0)$ to final value $x(t_f)$. In the right plot, the line segment is the optimal trajectory of the costate λ_2, its negative is shown. The projection onto the unit circle is the optimal control u. The projection is illustrated with dotted lines that appear like spokes on a wheel, each spoke connecting λ_2 to u at the same point in time.

control is the projection of the adjoint line segment onto the unit circle, which is the boundary of the admissible control set $\mathcal{U}$.

If λ_2 passes through the origin, the problem reduces to that of the double integrator in $\mathbf{R}^1$ with a bang-bang controller as the solution. In Fig. 1, this would be the case if the solid line in the right plot passed through the origin. The projection of the line segment onto the unit circle would then be two points, resulting in a control law that switches from one constant control to the diametrically opposite control at the appropriate switching time. This special case is adequately addressed in the literature; we do not further consider it here.

C. Solution by Shooting Method

Necessary conditions 3, 4, 5 and 6 define a two-point boundary value problem. That is, the differential equations have mixed initial and final conditions. Necessary condition 4 is vacuous for this problem since there is an explicit terminal state $\mathcal{X}_f = \{x_f\}$. Necessary condition 2 must be enforced on the two-point boundary value problem. Since the state dynamics are time invariant, it is sufficient only to enforce this condition at the final time:

$$H_f^* = H(x^*(t_f), u^*(t_f), \lambda^*(t_f)) = 0 \tag{12}$$

We use the forward shooting method described in Fotouhi-C. and Szyszkowski[2] to solve the two-point boundary value problem. Five optimal parameters are sought: the terminal time t_f and the initial costate $\lambda(0) = (\lambda_{10_t}, \lambda_{10_n}, \lambda_{20_t}, \lambda_{20_n})$. Once these parameters are found, the problem is solved. The optimal control u^* would then be given by Eq. (9), where $\lambda^*(t)$ would be found by numerically integrating the adjoint equations from $\lambda(0)$ to the terminal time t_f.

The sought-after parameters are solved by an iterative search. Given candidate t_f and $\lambda(0)$, forward differentiation of the state and adjoint equations determine $x(t_f)$, $u(t_f)$, and $\lambda(t_f)$, thus determining the terminal value of the Hamiltonian $H_f = H(x(t_f), u(t_f), \lambda(t_f))$. For this solution to be time optimal, the terminal state must satisfy $x(t_f) = x_f^* = (0, 0, x_{2f_t}, x_{2f_n})$ and the Hamiltonian must satisfy $H_f = 0$ from Eq. (12). Accordingly, define the vector function g by

$$g(t_f, \lambda(0)) = \begin{bmatrix} x(t_f) - x_f^* \\ H_f \end{bmatrix} \tag{13}$$

where t_f and $\lambda(0)$ are optimization parameters. By minimizing $\|g\|_2$, which we call the search index, the optimal t_f and $\lambda(0)$ are found and the problem is solved, but only if the search index is adequately close to zero.

The challenge is to come up with an intelligent guess of t_f and $\lambda(0)$ which starts the process. The search index typically possesses other minima and such minima may or may not be at zero. The shooting method can fail in several ways:

- It might converge to a nonzero minimum. This is not an optimal solution because the necessary conditions are not satisfied. Try a different guess of t_f and $\lambda(0)$.
- It might converge to a zero minimum that is not the global minimum. This satisfies the necessary conditions, hence is a local minimum. Other means, beyond the scope of this paper, should be used to verify that it is the desired global minimum. If it is determined that this is not the global minimum, try a different guess of t_f and $\lambda(0)$.
- It might not converge. Try a different guess of t_f and $\lambda(0)$.

Luck in choosing good initial search parameters t_f and $\lambda(0)$ is crucial to solving for the optimal control. Not only may the search fail as indicated above, but the search could expend significant computer time in blindly starting the search for the solution. We seek a means to intelligently choose the initial search parameters t_f and $\lambda(0)$.

D. Sensitivity of Shooting Method to Initial Guess

Figure 2 demonstrates the sensitivity of the shooting method to the initial guess of the optimization parameters. For each figure, a set of initial and final states were chosen and the optimal solution precomputed

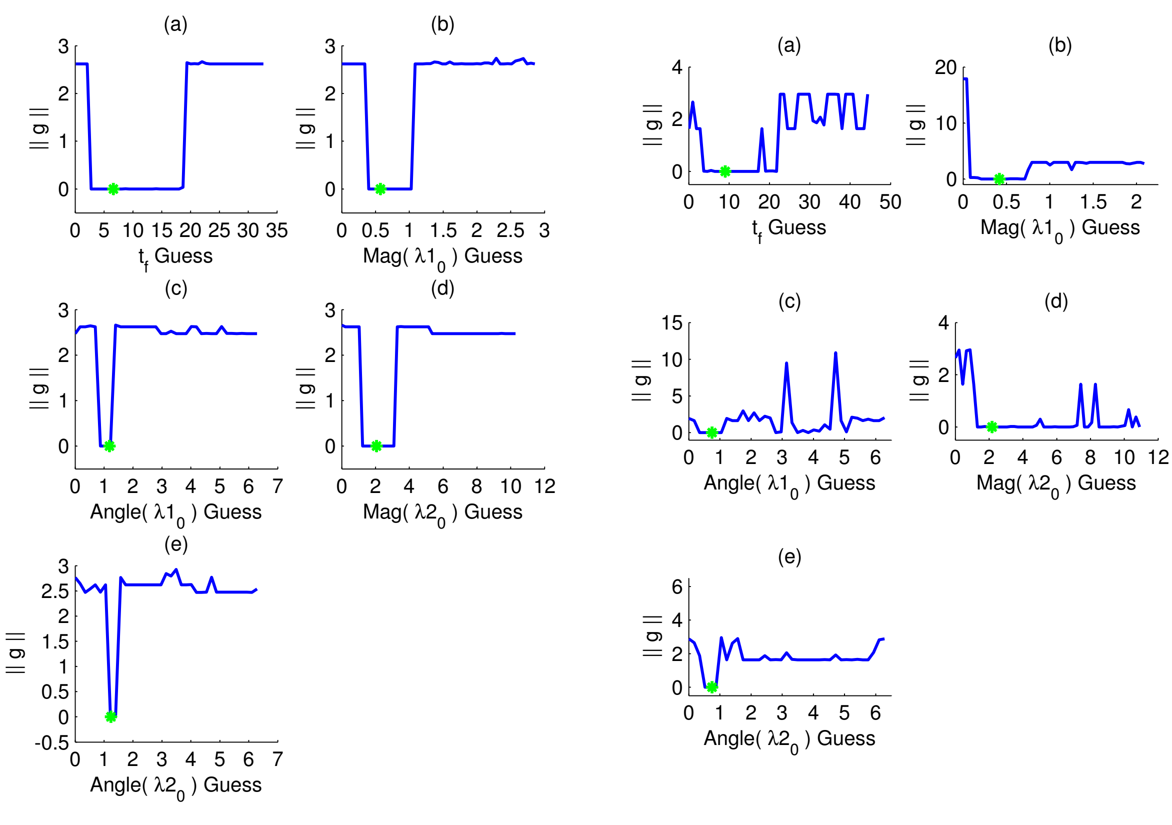


Figure 2. Sensitivity of shooting method to the initial guess of the optimization parameters t_f and $\lambda(0)$. In each case, a window of acceptable initial search parameters appears around the optimal search parameters denoted by a star. Convergence is assured when the initial guess for these parameters lies inside these windows.

along with the corresponding optimization parameters. This becomes the reference solution. The optimization parameters were then varied one at a time about the reference solution, used as the initial guess. The resulting search index $\|g\|$ was plotted against the optimization parameter.

In the left set of plots in Fig. 2, the initial condition is $x_0 = (1, 0, -1, -1)$ and the final condition is $x_f = (0, 0, -1, 1)$. For this fairly simple example, it is seen that each parameter has a range of values around the actual solution within which the shooting method will converge to the optimal solution. It is seen that the ability to find the optimal solution highly depends on the initial guess of the *directions* of the two costate vectors. Thus an intelligent algorithm must be very accurate in estimating the directions of the costate vectors.

In the right set of plots, the initial condition is $x_0 = (1, 0, 1, 1)$ and the final condition is $x_f = (0, 0, 1, 1)$. Although similar to the first set of initial and final conditions, the sensitivity to the initial guess is radically different, and yet, a window of appropriate initial conditions to use still exists around the true parameters. Moreover, the ability to converge to the optimal solution is still highly dependent on a good estimate of the directions of the costate vectors.

E. Intelligent Guess for Final Time and Initial Costates

Finding the optimal solution using the shooting method depends on an intelligent choice of the search parameters t_f and $\lambda(0)$. The relative simplicity of the solution for the double integrator in $\mathbf{R}^2$ allows one to estimate these parameters using the geometry of the adjoint solution as it relates to the optimal control. Envisioning the geometry is easy.

We now borrow the solution from another optimal control problem to help mathematically implement the geometric guess. The dual to minimum time/constrained control is minimum energy/unconstrained control. An analytic solution for $\lambda(0)$ as a function of t_f for the minimum energy/unconstrained control double integrator in $\mathbf{R}^1$ exists[5] and has been extended to the $\mathbf{R}^2$ case:

$$\lambda_1(0) = -\frac{12}{t_f^3}(x_1(t_f) - x_1(0)) + \frac{6}{t_f^2}(x_2(t_f) + x_2(0)) \tag{14}$$

$$\lambda_2(0) = -\frac{6}{t_f^2}(x_1(t_f) - x_1(0)) + \frac{2}{t_f}(x_2(t_f) + 2x_2(0)) \tag{15}$$

The optimal control for the fuel-optimal problem is $u^* = -\lambda_2$, while that for the time-optimal problem in Eq. (9) is very similar. We conjectured that using the initial conditions for the fuel-optimal problem would be sufficient to initiate the solution of the time-optimal problem. The conjecture is valid, except a good estimate of the final time is required.

Estimation of the final time t_f is now investigated. Figure 3 demonstrates the sensitivity of the shooting method to the estimate of the final time. The same set of initial and final conditions were chosen as for the example in Fig. 2 and the estimate of t_f was varied. The initial costate was estimated using Eqs. (14)

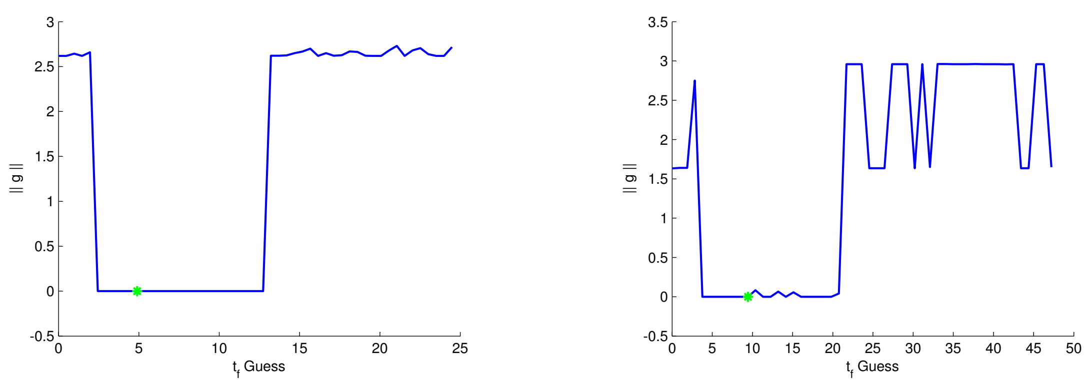


Figure 3. Sensitivity of the shooting method to the estimate of the final time t_f.

and (15). The search index $\|g\|$ was calculated and plotted against t_f. It is seen that the convergence of the shooting method is fairly robust to changes in the final time estimate. The problem is now reduced to determining a good estimate of only one parameter, the final time.

To find an estimate for the final time, assume temporarily that the optimal control is a constant $\bar{u}$. Integrating the equations of motion, $\ddot{x}_1 = \bar{u}$, twice and constraining the final velocity as $x_{2f} = 0$, the final time can be directly calculated as:

$$t_f = -\frac{v_t}{\bar{u}_t} + 2\sqrt{\frac{|v_t^2 - 2\bar{u}_t d|}{2\bar{u}_t^2}} \tag{16}$$

where

$$x_1(0) = \begin{bmatrix} d \\ 0 \end{bmatrix}, \qquad x_2(0) = \begin{bmatrix} v_t \\ v_n \end{bmatrix}, \qquad \bar{u} = \begin{bmatrix} \bar{u}_t \\ \bar{u}_n \end{bmatrix}. \tag{17}$$

Given a good estimate of the average control input $\bar{u}$ for the optimal trajectory, the time required to reach the final state can thus be estimated. To get an estimate for $\bar{u}$, two sets of initial conditions are used in which the magnitude of the control is easily calculated. A weighted average of the estimates of $\bar{u}$ is then used to obtain the true estimate.

For the first set, assume that the initial horizontal velocity is zero ($v_t = 0$). For this example, the time required to drive the horizontal position to rest at zero is

$$T_h = 2\sqrt{\left|\frac{d}{\bar{u}_t}\right|}. \tag{18}$$

In the vertical direction, the initial velocity is not zero, but the position is zero. The time required to slow down the vertical motion and return the vertical position to rest at zero is computed as

$$T_v = (1 + \sqrt{2})\left|\frac{v_n}{\bar{u}_n}\right|. \tag{19}$$

Since both states are required to return to the origin at the same time, Eqs. (18) and (19) can be equated. Using the constraint on the control that $\|u\|_2 \leq u_{max}$, the components of $\bar{u}$ can be calculated as

$$\bar{u}_{t1} = -\frac{(3 + 2\sqrt{2})v_n^2}{8d} + \frac{1}{2}\sqrt{\frac{(17 + 12\sqrt{2})v_n^4}{d^2} + 4u_{max}^2} \tag{20}$$

$$\bar{u}_{n1} = \sqrt{u_{max}^2 - \bar{u}_{t1}^2} \tag{21}$$

For the second set, assume that the initial horizontal position is zero, and the initial horizontal velocity is non-zero. In a similar fashion, the horizontal component of $\bar{u}$ is calculated to be

$$T_h = T_v \tag{22}$$

$$(1 + \sqrt{2})\left|\frac{v_t}{\bar{u}_t}\right| = (1 + \sqrt{2})\left|\frac{v_n}{\bar{u}_n}\right| \tag{23}$$

$$\bar{u}_{t2} = u_{max}\sqrt{\frac{v_t^2}{v_t^2 + v_n^2}} \tag{24}$$

$$\bar{u}_{n2} = \sqrt{u_{max}^2 - \bar{u}_{t2}^2} \tag{25}$$

If the initial conditions match those given by either of the two sets, $\bar{u}$ is given by that value calculated for the respective set. However, in the general case where both the initial horizontal position and velocity are non-zero, the average control $\bar{u}$ can be computed using a weighted average of the calculated values for the two sets:

$$\begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{bmatrix} = |d| \begin{bmatrix} \mathrm{sgn}\left(\frac{d}{v_t}\right)\bar{u}_{t1} \\ \bar{u}_{n1} \end{bmatrix} + |v_t| \begin{bmatrix} \bar{u}_{t2} \\ \bar{u}_{n2} \end{bmatrix} \tag{26}$$

$$\bar{u} = \frac{u_{max}}{\bar{u}_1^2 + \bar{u}_2^2} \begin{bmatrix} \bar{u}_1 \\ -\mathrm{sgn}(v_t)\,|\bar{u}_2| \end{bmatrix} \tag{27}$$

F. Simulation Results

The effectiveness of the final time estimation algorithm is shown in Figs. 4 and 5, where one of the state parameters was varied in each set of subplots (a) – (f). The estimate of the final time and the initial costate vector was calculated using the above method. The shooting method was then used to find the optimal trajectory with the provided estimates. The search index $\|g\|$ was calculated and plotted in subplot (a) against the varying parameter to confirm that the shooting method converged to the optimal trajectory. In subplots (b) – (f), the guess of the optimization parameters from the above algorithm and the actual values found from the shooting method are plotted against the varying state parameter. The final time parameter is plotted in subplot (b), the magnitude of costate λ_{10} is in subplot (c), the angle from horizontal of the costate λ_{10} is in subplot (d), the magnitude of costate λ_{20} is in subplot (e), and the angle from horizontal of the costate λ_{20} is in subplot (f).

On the left side of Fig. 4, the state parameter that was varied was the initial distance from the origin, d. The other states are given by $v_n = 1, v_t = 0$, and $x_f = (0, 0, 0, 0)$. On the right side, the varying parameter was the initial vertical velocity, v_n, with the other parameters given by $d = 1, v_t = 0$, and $x_f = (0, 0, 0, 0)$.

On the left side of Fig. 5, the varying state parameter was the initial horizontal velocity, v_t, while the other states are given by $d = 1, v_n = 1$, and $x_f = (0, 0, 0, 0)$. On the right side, the varying state parameter was the final velocity, x_{2f}, with the other states given by $d = 1, v_n = 1, v_t = 0$ and $x_{1f} = (0, 0)$.

In all cases, the estimate of the direction of the initial adjoint vectors is very good, enabling the shooting method to converge to the optimal solution for each set of state conditions. In general, the final time estimate is also very good, good enough to ensure convergence to the optimal solution. But it also suggests that it does better than this: the algorithm itself finds a nearly optimal solution.

The estimate of the initial magnitudes of the costate vectors is not as accurate. It is suspected that the adjoint estimation equations from Junkins,[5] Eqs. (14) and (15), can be modified to get closer to the optimal estimate, but this is beyond the scope of this paper. A better estimate of the magnitudes should also speed up the convergence time of the shooting method. By extending the algorithm to obtain a better estimate of the magnitudes of the costate vectors and to provide a better estimate of the final time for large final

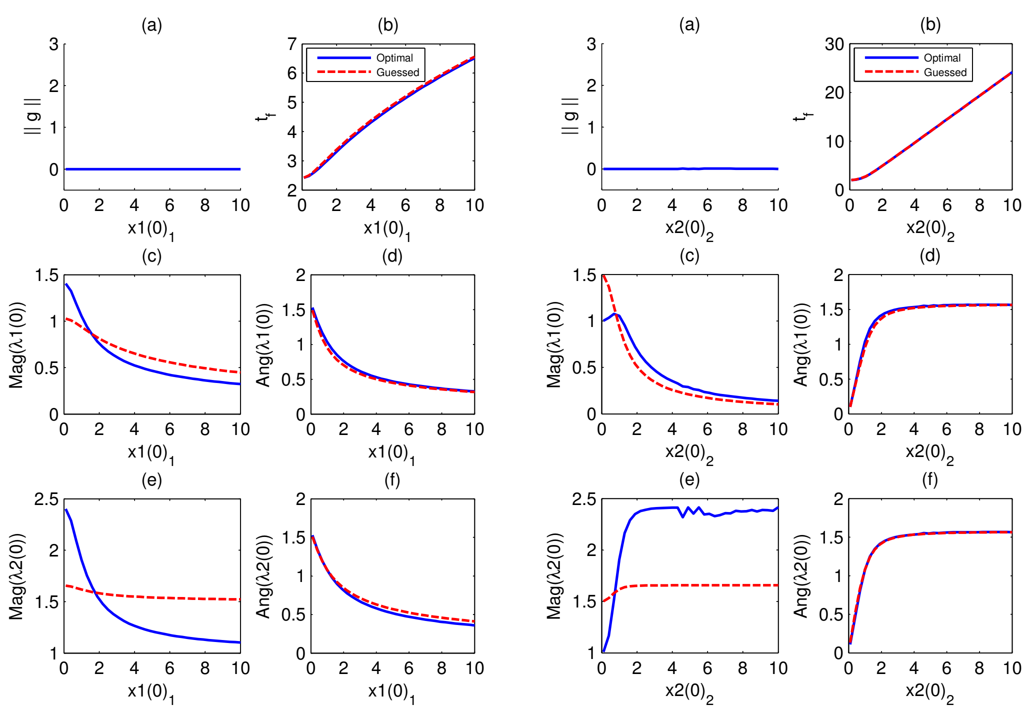


Figure 4. The effectiveness of the final time estimation algorithm for two sets of state parameters.

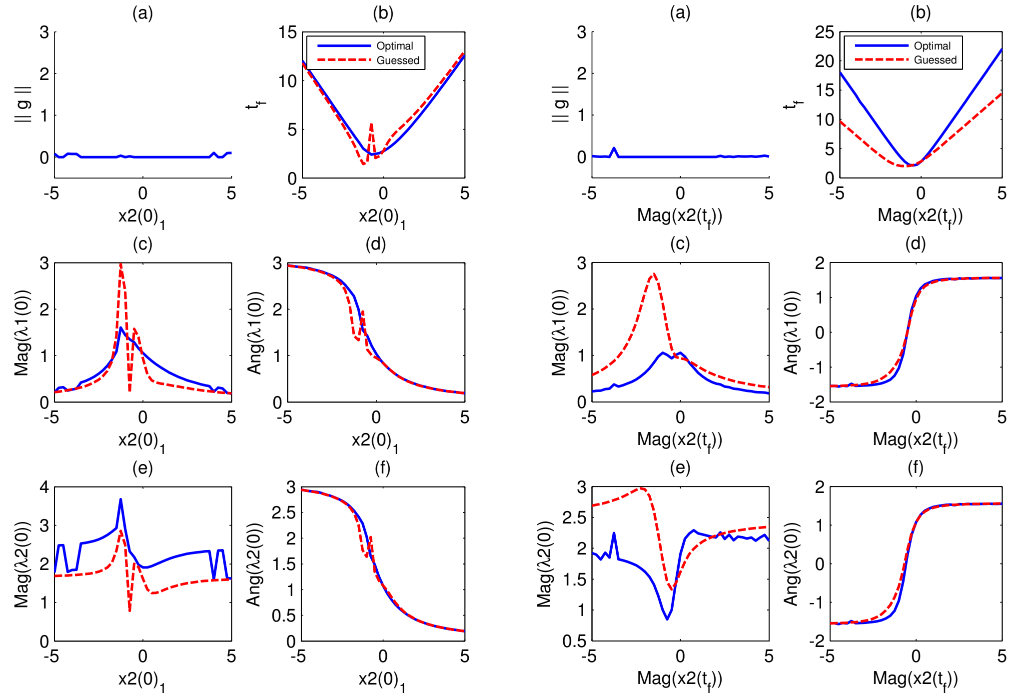


Figure 5. The effectiveness of the final time estimation algorithm for two more sets of state parameters.

velocities, the possibility exists that the algorithm can find the optimal trajectory using only the geometry of the problem.

Figure 6 shows the optimal trajectories for varying initial and final conditions. In each plot, a parameter θ was varied and is defined as follows. The plot on the left corresponds to initial condition $d = 1, v_t = \cos\theta, v_n = \sin\theta$ and final condition $x_f = (0, 0, 0, 0)$. The plot in the middle depicts initial condition $d = 1, v_n = 0.5, v_t = 0$, $x_1(0) = (\cos\theta, \sin\theta)$, and $x_f = (0, 0, 0, 0)$. The plot on the right shows trajectories corresponding to initial condition $d = 1, x(0) = (\cos\theta, \sin\theta, 0, 1)$ and final condition $x_f = (0, 0, 1, 0)$. These trajectories demonstrate the success of the algorithm in finding the optimal trajectory, and the ability of the trajectory to be rotated back into the original frame, allowing for all problems in R^2 to be solved.

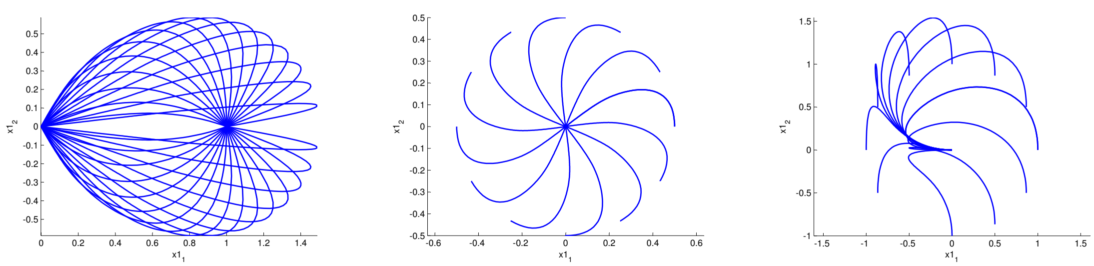


Figure 6. The effectiveness of the final time estimation algorithm. Optimal trajectories are shown for varying state conditions.

V. Orbit Transfer in $\mathbf{R}^2$

The necessary conditions for time-optimal control are now applied to the transfer of a spacecraft from one orbit to another. A spacecraft initially in one circular orbit of radius r_0 is to be transferred by its own limited thrust to a higher (or lower) circular orbit of radius r_f in minimum time. In-plane transfer is assumed, so this becomes a two-dimensional problem. The adjoint equations are formed and solved numerically which result in the time-optimal control law. Shooting methods must again be used because the problem reduces to a two-point boundary value problem.

Despite being completely different physical problems, there are striking similarities between the double integrator and orbit transfer problems when the layers of optimality are peeled back. When viewed from the right vantage, the adjoint equations and optimal control for the orbit transfer problem are remarkably similar to that for the double integrator. This similarity is exploited by borrowing the double integrator algorithm that intelligently selects initialization parameters for the shooting algorithm, and applying it to the orbital transfer problem.

The optimal solution is demonstrated in a numerical example. The orbit raising problem was chosen mostly because it is about the simplest problem in orbital mechanics, and useful at the same time, but also because it is notoriously difficult to guess the initial parameters required to start the shooting algorithm. The search space is littered with false minima. These are partly due to the "circular wrap" characteristic of planetary orbits, but also because there are plenty of hiding places where local minima might lie with a search parameter space of five dimensions.

A. Problem Definition

Consider the motion of a spacecraft orbiting a large body with gravitational parameter μ. Under appropriate simplifying assumptions, the equations of motion are given by

$$\ddot{r} = -\frac{\mu}{\|r\|^3} r + u, \qquad r(0) = r_0 \tag{28}$$

where $r(t) \in \mathbf{R}^2$ for all $t \geq 0$. To understand the parallels between this problem and the double integrator, we use the same notation for state developed in the previous section. This means we replace r with x_1 and $\dot{r}$ with x_2 so the phase-canonical form of the state is

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbf{R}^4. \tag{29}$$

Just as for the double integrator, the thrust control $u(t) \in \mathbf{R}^2$ is admissible if $u_1^2(t) + u_2^2(t) \leq u_{max}^2$ for all $t \geq 0$.

B. Application of Maximum Principle

We now apply Pontryagin's Maximum Principle. The Hamiltonian is identical in form to that of the double integrator as it appears in Eq. (8):

$$H = 1 + \lambda_1^{\mathrm{T}} x_2 + \lambda_2^{\mathrm{T}} u \tag{30}$$

where as before the costates are $\lambda_1(t), \lambda_2(t) \in \mathbf{R}^2$ for all $t \geq 0$.

From necessary condition 1, the optimal control is also identical in form to that of the double integrator:

$$u^*(t) = -\frac{\lambda_2(t)}{\|\lambda_2(t)\|_2} u_{max} \tag{31}$$

The exact similarity to the double integrator ends when one forms the adjoint equations:

$$\frac{d}{dt}\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 & \Gamma \\ -I & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \tag{32}$$

where I is the 2×2 identity matrix and Γ is the gravity gradient matrix

$$\Gamma = \frac{\mu}{r^3} \begin{bmatrix} 1 - 3\cos^2\theta & -3\sin\theta\cos\theta \\ -3\sin\theta\cos\theta & 1 - 3\sin^2\theta \end{bmatrix}. \tag{33}$$

Here, θ is the true anomaly. Note that $\Gamma = 0$ when $\mu = 0$, thus the orbital transfer problem is a bifurcation of the parameter μ; the problem reduces to that of the double integrator when $\mu = 0$. In this case, note that both the state and adjoint equations reduce to that of the double integrator given in Eqs. (5) and (10).

The solution to the adjoint equations is not a simple straight line, as was the case for the double integrator, but they do form a set of linear differential equations. When transformed to local level, the gravity gradient matrix is

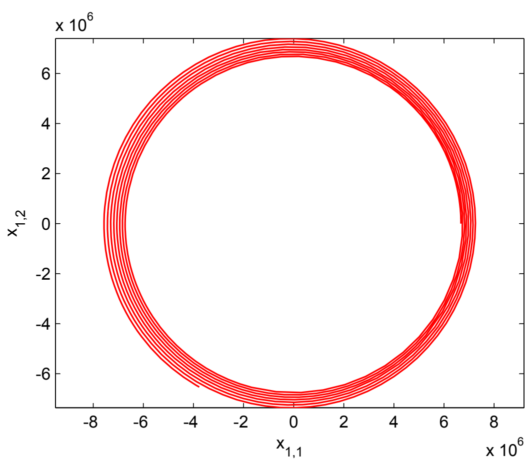
$$\Gamma = \frac{\mu}{r^3} \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \tag{34}$$

so for small orbit changes, the adjoint equations are nearly time invariant. In fact, there is a closed form solution to the adjoint equations when r is constant, and when transformed to local level it is given by a linear combination of a pair of sinusoids at the Schuler frequency, $\omega_S = \sqrt{\mu/r^3}$, in the local horizontal direction and a pair of complementary exponentials in the vertical direction, typical of a Hamiltonian system.

Similar to the double integrator, the optimal controller is the projection of the costate variable λ_2 onto the circle with center at the origin and radius u_{max}. For small orbit changes, the analytic solution of the adjoint equations can be projected onto this circle to infer the optimal control. Starting conditions for the search parameters of the forward shooting method can be estimated from this geometry, similar to that for the double integrator. A preliminary look was taken at how this would be accomplished, but a full algorithm is not in place. Instead, results are given in the next section that were generated using engineering information gleaned from the geometry. It appears a full algorithm can be developed, and this will be reported on in the future.

C. Simulation Results

A time-optimal orbit raising maneuver is illustrated in Fig. 7. A spacecraft orbiting the Earth is in a circular orbit of altitude 300 km, then transfers to a higher circular orbit of radius of 1100 km in minimum time. The thrust magnitude is limited to 1 cm/s^2. The true anomaly at the completion of the maneuver is not important, so it is not specified. There are thus only three terminal state conditions: the spacecraft be at



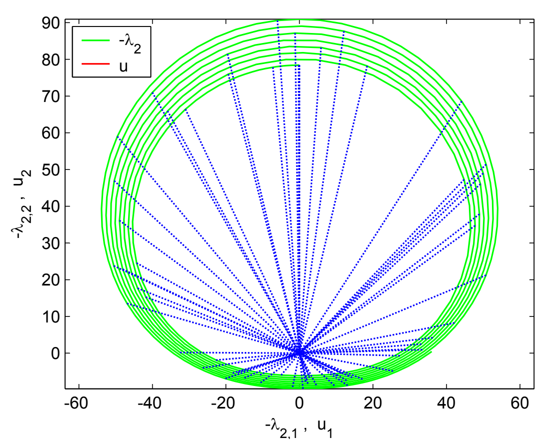


Figure 7. Minimum time orbit transfer from 300 km circular orbit to 1100 km circular orbit, with maximum thrust magnitude of 1 cm/s^2. The minimum time to raise the orbit is just over 12 hr. The left plot shows the trajectory of the spacecraft. The right plot shows the adjoint solution (large circles) and the optimal control (tiny circle). The projection of the adjoint onto the circle is illustrated with dotted lines that appear like spokes on a wheel, each spoke connecting λ_2 to u at the same point in time.

the specified radius and its velocity be that of a circular orbit. Mathematically, the terminal manifold is given by

$$\mathcal{X}_f = \{(x_1, x_2) \in \mathbf{R}^4 : \|x_1\|_2 = r_f, \quad x_1 \cdot x_2 = 0, \quad \|x_2\|_2 = \sqrt{\frac{\mu}{r_f}}\} \tag{35}$$

In this case, the transversality condition[4] was applied to replace the non-condition on the true anomaly. The formulation of the transversality condition is standard, with details omitted.

In the left plot of Fig. 7, the trajectory is close to a logarithmic spiral and it is also close to that when thrusting fully in the velocity direction, but the minimum time is less here as compared to both of these cases.

Some but not all of the elements of the algorithm for initiating the search parameters used in the double integrator were used here. For example, the initial costate λ_2 must be very close in direction to the negative of the velocity vector, as it would be for a spacecraft maximizing increase in orbital energy. It was also learned that a good estimate of the terminal time t_f is paramount to starting the shooting algorithm, just as it was for the double integrator.

VI. Conclusion

Pontryagin's Maximum Principle was applied to the time-optimal control of two problems, the double integrator and the orbit transfer of a spacecraft. Although the two problems appear very different, they have remarkably similar optimality structure. The two-point boundary value problem that results from applying the principle was solved numerically, using forward shooting methods. The challenge in using shooting methods is determining good guesses of the initial costate and terminal time. An algorithm was developed for this purpose for the double integrator and it was found to work quite well. Elements of the algorithm were applied to the thrust control of a spacecraft transferring from one circular orbit to another in minimum time.

Future directions include the following:

- Initializing the shooting algorithm for the orbit transfer problem is notoriously difficult. An algorithm for estimating the initial search parameters should be developed. A possible starting point would be the algorithm used for the double integrator developed in this paper.
- The use of backward, rather than forward, integration methods to solve the two-point boundary value problem should be favored. Because of duality, adjoint equations naturally run backwards in time. When tranversality is imposed, the dimension of the search space is reduced when integrated backwards.
- The double integrator on $\mathbf{R}^2$ can describe the dynamics on other manifolds, such as $\mathbf{S}^2$. On $\mathbf{S}^2$, the control of a double integrator is through torque, not force. This suggests that applications related to time-optimal attitude control might benefit from the ideas developed in this paper.

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