Optimal Waypoint Scheduling of an Imaging Satellite

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BIOGRAPHIES

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ABSTRACT

A new result in optimal control is applied to the scheduling of high-resolution imaging of successive ground targets from an orbiting satellite. In the envisioned low-cost satellite application, attitude control is accomplished solely through reaction wheels; there is no thrusting capability. Reaction wheels are particularly suited to pointing applications in which the spacecraft rotates slowly or not at all. To quickly change the pointing direction to a new target, either the reaction wheels must be powerful or the satellite moment of inertia must be small. Design tradeoffs dictate that slew time between targets is not insignificant, reducing the economic value per orbit of the mission. Solving for the optimal solution is complicated by the fact that attitude control dynamics are nonlinear (unit quaternions for attitude and Euler equations for attitude rate), there is limited torque magnitude available to the reaction wheels, and there is the need to periodically dump momentum. The objective of this study is to determine time optimal control policies to slew between scheduled waypoint views. The multiple-interval generalization of Pontryagin´s Maximum Principle, established in a soon-to-be-complete PhD dissertation, is proposed to find the optimal attitude policy. The generalization addresses the total scheduling problem over a given orbit, not just time minimization from waypoint to waypoint. An example based on design characteristics of a small imaging satellite is set up, and solution methods are explored.

INTRODUCTION

Pontryagin's Maximum Principle was developed in the 1950's in response to rapid advances made in missile technology. As the Cold War escalated, it became possible to deliver missiles long distances with reasonable accuracies. A theory was needed on how to deliver the missiles optimally. The less time it takes for a missile to arrive at its target, the more military value it has; the less fuel it expends, the lighter it can be hence increasing target range.

Lev Pontryagin (1908-1988) and his assistants solved the problem in the 1950's, providing the theoretical framework for this and other similar problems [1]. The theory prescribes a well-defined set of differential equations and boundary conditions which generally leads to a solution that both exists and is unique.

Pontryagin's Maximum Principle is a collection of necessary conditions for optimal control that best transfers a linear or nonlinear dynamical system from one state to another. The principle accommodates state and control constraints. It is a variational method that identifies local extrema, but frequently it finds the global extreme. The principle is closely related to the calculus of variations, the method of Lagrange multipliers, and the Karush–Kuhn–Tucker conditions. The main power of the principle is that it reduces an infinite-dimensional function space problem to one of finite dimensions.

Recently, a new result in the optimal control of nonlinear systems was developed by the authors [2]. The new result extends Pontryagin's Maximum Principle to apply to multiple intervals. The generalization also accommodates state constraint interdependencies and parametric optimization. From interval to interval, everything about the problem can be changed, including the differential equations, the state size, the set of admissible controls, the performance criterion, and the boundary conditions.

Some portions of the new result appear in previous works, such as interior point equality constraints [3]. For the most part, however, they have been applied by appeal to the intuition, rather than by rigorous proof. We do not dispute their validity and, in fact, our theorem legitimizes their use and extends their applicability considerably.

Our new result unifies theory across several fields and has diverse applications. It allows for periodic control as well as network optimization, as illustrated in Fig. 1.

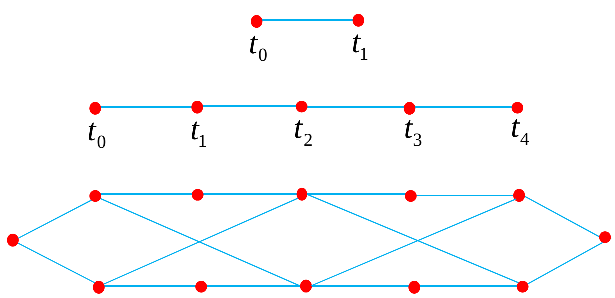

Figure 1. The top subfigure represents single-interval control. State boundary conditions are represented by the red dots and differential equations by the blue lines. The new result generalizes this to multiple intervals, pictured in the middle subfigure. The bottom subfigure represents a network that can be optimized using the new result.

Applications of the new result include:

- Optimal steering of vehicles through waypoints,
- Optimal grand tour of the solar system using planetary gravity assists,
- Optimal periodic control such as transoceanic flight of an albatross or repetitive tasks in automated manufacturing,
- Generalized spline interpolation,
- Optimal systems of partial differential equations, and
- Optimal base running for inside the park home run.

In this paper, we apply the new theory to optimally schedule an imaging satellite to point at successive ground targets. The scheduling is optimal with respect to time or energy, or some weighted combination thereof determined by the designer. The equations of motion are nonlinear, which the theory accommodates. The goal of the paper is to report the new result in context with the satellite pointing application, set up the multipoint boundary value problem implied by the new result, and discuss possible ways to solve the resulting equations.

PONTRYAGIN'S MAXIMUM PRINCIPLE

Consider the system of differential equations

$$\dot{x} = f(x, u) \tag{1}$$

defined on the interval $I = [t_0, t_f]$, where $x : I \to R^n$ is the state and $u : I \to R^m$ is the control. u is an *admissible control* if it is Lebesgue measurable, bounded on I, and $u(t) \in U$ for all t. U is an arbitrary specified subset of R^m. When u is an admissible control and f is a sufficiently smooth function of x and u, (1) generally has a unique solution on interval I for any given initial condition.

Solutions x of (1) are deemed *feasible* if there exists an admissible control u that drives the state from some initial condition $x(t_0) \in M_0$ to some final condition $x(t_\mathrm{f}) \in M_\mathrm{f}$, where M_0 and M_f are given smooth manifolds in R^n. A manifold is simply a smooth surface in R^n, an example of which is the unit sphere in R^3. A manifold is typically defined by a set of k scalar algebraic equations, in which case the manifold is said to have dimension $n - k$. For example, the unit sphere in R^3 is defined by the single equation $x^2 + y^2 + z^2 = 1$, so it is a 2-dimensional manifold. The introduction of M_0 and M_f allow us to accommodate incompletely specified state in a fairly general manner. If the above conditions are satisfied, we say that (x, u) is a *feasible pair*.

Consider feasible pairs (x, u), when they exist, that minimize

$$J(x, u) = \int_{t_0}^{t_\mathrm{f}} f^0\,(x(t), u(t))dt \tag{2}$$

where f^0 is a sufficiently smooth function of x and u. In this case, (x, u) is said to be an *optimal pair*. When an optimal pair (x, u) exists, it attains the global minimum of $J(x, u)$ over all feasible pairs, but it may not be unique. Even if feasible pairs exist, there is no guarantee that any are optimal.

The Hamiltonian is defined as the scalar function

$$H(x,u,\lambda^0,\lambda) = \lambda^0 f^0(x,u) + \lambda^{\mathrm{T}} f(x,u)$$

where $\lambda : I \to R^n$ is called the *costate*. In this and other similar applications, we can set the scalar $\lambda^0 = 1$ without significant loss of generalization. Doing so simplifies the Hamiltonian to

$$H(x,u,\lambda) = f^0(x,u) + \lambda^{\mathrm{T}} f(x,u) \qquad (3)$$

which we use henceforth. Roughly speaking, the Hamiltonian is the infinite-dimensional analogue of adjoining a finite dimensional constrained objective function with Lagrange multipliers.

Transversality is now defined. Let M be a manifold in R^n. The tangent plane of a manifold M at a point $x \in M$ is denoted by $T_x M$. A vector $\lambda \in R^n$ satisfies the *transversality condition* with respect to M at x if $\lambda^T \mu = 0$ for all $\mu \in T_x M$. The condition is equivalently stated as $\lambda \perp T_x M$. When manifold M consists of a single point, for example, the transversality condition is vacuous. At the other extreme, when $M = R^n$, the transversality condition requires $\lambda = 0$.

A version of the original Pontryagin's Maximum Principle useful for our purposes is now stated. Some of the more esoteric technical conditions are omitted for clarity, and can be found in [1].

Theorem 1. Let $\dot{x} = f(x,u)$ be defined on the interval $I = [t_0, t_{\mathrm{f}}]$ and suppose M_0 and M_{f} are manifolds that encode initial and final conditions, respectively. If (x,u) is an optimal pair, then there exists an absolutely continuous function $\lambda : I \to R^n$ such that

1. The adjoint equations $\dot{\lambda} = -\frac{\partial H}{\partial x}$ hold a.e. on I,
2. The minimum condition
$$H(x(t),u(t),\lambda(t)) = \min_{v \in U} H(x(t),v,\lambda(t))$$
holds for almost every $t \in I$,
3. $\lambda(t_0)$ satisfies the transversality condition with respect to M_0 at $x(t_0)$ and $\lambda(t_{\mathrm{f}})$ satisfies the transversality condition with respect to M_{f} at $x(t_0)$,
4. When the times t_0 and t_{f} are free,
$$H(x(t_{\mathrm{f}}),u(t_{\mathrm{f}}),\lambda(t_{\mathrm{f}})) = 0.$$

Furthermore, for any absolutely continuous function λ that satisfies conditions 1 and 2, the time function $H(x(t),u(t),\lambda(t))$ is constant on I.

Application of Theorem 1 reveals that there are $2n$ first-order differential equations, n state equations and n adjoint (or costate) equations. To solve them, $2n$ boundary conditions are needed, no more and no less. When the initial and final states are specified, this creates the required $2n$ conditions. When either or both the initial or final state are not fully specified (i.e., some states are left free to be optimized), there are less than $2n$ state boundary conditions. The remaining boundary conditions "migrate" one-for-one to costate boundary conditions through transversality. The result is a two-point boundary value problem with the number of first-order differential equations precisely equal to the number of boundary conditions.

As mentioned in the Introduction, the usefulness of Pontryagin's Maximum Principle is that it reduces an infinite dimensional optimization problem to a finite dimensional problem. For well-posed problems, application of these necessary conditions typically identifies a single feasible pair (x,u) which is the only possible optimal solution. Theorem 1 does not guarantee that this solution is optimal, but it does guarantee either that it is optimal or that no optimal solution exists. Other means, usually knowledge of the application, is then used to resolve between the two possibilities.

Two examples demonstrating Pontryagin's Maximum Principle are illustrated in Figs. 2 and 3.

MULTIPLE INTERVAL EXTENSION OF PONTRYAGIN'S MAXIMUM PRINCIPLE

In [2], we generalize Pontryagin's Maximum Principle to apply to an interval $I = [t_0, t_f]$ partitioned into a grid of knots $t_0 < t_1 < \cdots < t_K = t_f$. The knots can be fixed or free. Denote the closed subintervals of the grid by $I_k = [t_{k-1}, t_k]$ for $k = 1, \ldots, K$. Constraints, such as interior point equality constraints (waypoints), can be applied at the knots. As mentioned in the Introduction, system characteristics can also differ from subinterval to subinterval. In this paper, we demonstrate the multipoint

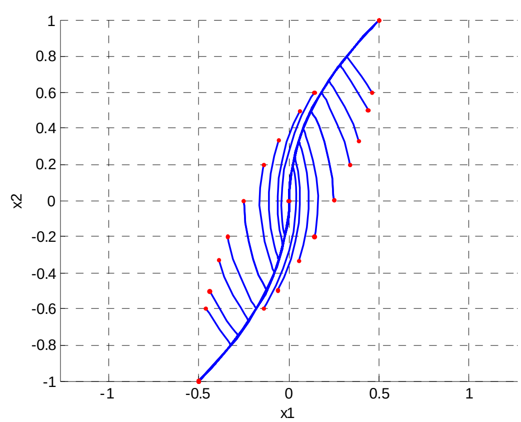

Figure 2. Time optimal control of a double integrator. Phase plane shows optimal trajectory, x_1 is position and x_2 is velocity. The red dots represent various initial conditions. The optimal control is $u_{\max}$ until reaching switching curve, then it jumps to $u_{\min}$ driving the state to the origin in minimum time. Pontryagin formalized the theory behind bang-bang control, and extended it to many other important problems.

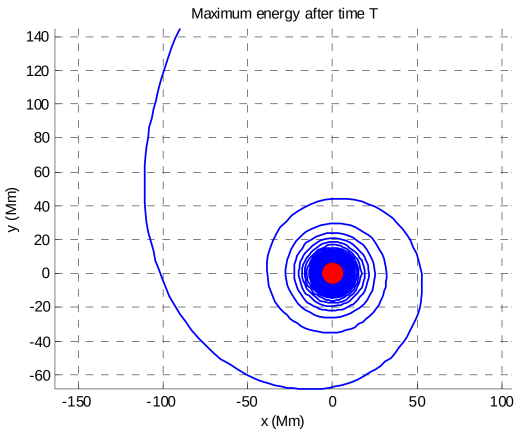

Figure 3. Optimal thrust control of spacecraft to reach escape velocity. Constant thrust ion engine powers spacecraft to maximize orbital energy after time T. Related problems that can be solved using Pontryagin's Maximum Principle are minimum time to escape velocity, minimum fuel to escape, and minimum constant thrust to escape in time T.

application without changing system characteristics across the subintervals.

Our new result affirms formation of the Hamiltonian of Theorem 1 from which both the adjoint equations and minimum condition follow. What is new is how the transversality condition is applied. We now state the multiple interval extension of Pontryagin's Maximum Principle in a form useful for the next section.

Theorem 2. Let $\dot{x} = f(x, u)$ be defined on the interval $I = [t_0, t_f]$ which is partitioned into a grid of knots $t_0 < t_1 < \cdots < t_K = t_f$. Also let $I_k = [t_{k-1}, t_k]$ for $k = 1, \ldots, K$. Suppose $\{M_k\}_0^K$ are manifolds into which boundary conditions at the knots have been encoded. If (x, u) is an optimal pair, then there exists a function $\lambda : I \to R^n$ that is absolutely continuous everywhere except possibly at the knots such that

1. The adjoint equations $\dot{\lambda} = -\frac{\partial H}{\partial x}$ hold a.e. on I,
2. The minimum condition
$$H(x(t), u(t), \lambda(t)) = \min_{v \in U} H(x(t), v, \lambda(t))$$
holds for almost every $t \in I$,
3. The transversality conditions are satisfied:
$$(\lambda(t_0), -\lambda(t_1^-), \lambda(t_1^+), -\lambda(t_2^-), \ldots, \lambda(t_{K-1}^+), -\lambda(t_K))$$
is orthogonal to the tangent space of M at the point
$$(x(t_0), x(t_1^-), x(t_1^+), x(t_2^-), \ldots, x(t_{K-1}^+), x(t_K))$$
where
$$M = M_0 \times M_1 \times M_1 \times M_2 \times \cdots \times M_{K-1} \times M_K,$$
and
4. When all the knots t_k are free,
$$H(x(t_k^-), u(t_k^-), \lambda(t_k^-)) = 0 \text{ for } k = 1, \ldots, K.$$

Furthermore, for any absolutely continuous (except possibly at the knots) function λ that satisfies Conditions 1 and 2, the time function $H(x(t), u(t), \lambda(t))$ is constant on each I_k.

Application of Theorem 2 reveals $2nK$ first-order differential equations (n state equations plus n adjoint equations, on each of the K intervals). To solve them, exactly $2nK$ boundary conditions are needed. When the initial and final states are specified on each subinterval I_K, this creates the required $2nK$ conditions. But then Theorem 2 offers nothing new, since Theorem 1 can be applied K times to obtain the optimal solution.

The power of Theorem 2 arises with incompletely specified state constraints, for example when position, but not velocity, is specified at the knots. For incompletely specified states, the theorem provides a recipe for constructing the $2nK$ boundary conditions. As before, the boundary conditions associated with unspecified states "migrate" one-for-one to costate boundary conditions through transversality. For the unspecified states, continuity of state must be enforced. It is noteworthy that Theorem 1 accomplishes this simply by "migrating" continuity of state to continuity of costate. Again, Theorem 1 prescribes exactly $2nK$ boundary conditions. The result is a multipoint boundary value problem with the number of first-order differential equations precisely equal to the number of boundary conditions.

Figure 4 shows an aircraft pylon race optimized using the new result, an example of optimal waypoint steering.

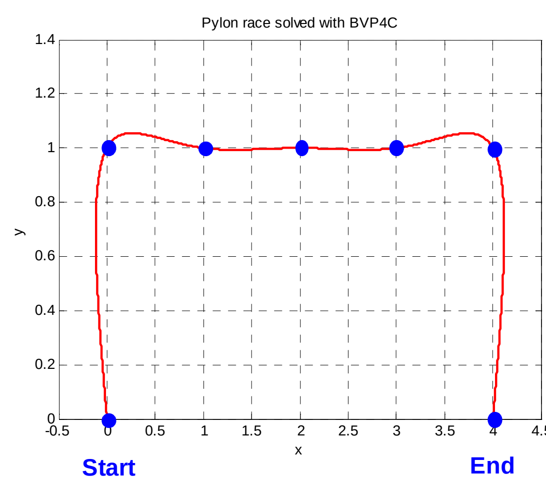

Figure 4. Aircraft races from pylon to pylon to minimize time on course. Aircraft has limited thrust, but can steer in any direction. In this multipoint optimization, there are seven waypoints, the pylons, and six subintervals ($K = 6$). Velocities at all waypoints, including start and end, are unspecified.

SATELLITE POINTING OPTIMIZATION

Theorem 2 is now applied to optimize the waypoint scheduling of sequentially pointing a satellite at multiple ground targets. The camera boresight direction slews between the ground targets, and reaction wheels are used to effect the required control torque. Because of limited control torque and nonzero moment of inertia of the satellite, it takes time to slew between targets. Furthermore, the targets are time-varying because the satellite is orbiting a rotating Earth, a fact at first we ignore in our initial attempt at solution.

We choose to minimize a weighted combination of energy expended and elapsed time according to a performance criterion given by

$$J = \int_{t_0}^{t_f} \{\gamma + u_x^2 + u_y^2 + u_z^2\}\, dt\,. \tag{4}$$

The parameter $\gamma \geq 0$ weighs the criterion between energy optimal (small γ) and time optimal (large γ), and takes on a fixed value assigned by the designer.

The equations of motion of satellite attitude are modeled in the following 7-state differential equation with unit quaternions used for attitude and Euler equations for attitude rate:

$$\dot{q} = \tfrac{1}{2}\Omega(\omega)q \tag{5a}$$

$$I\dot{\omega} + \omega \times (I\omega) = u \tag{5b}$$

where

$$q = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}, \quad \Omega(\omega) = \begin{bmatrix} 0 & -\omega_x & -\omega_y & -\omega_z \\ \omega_x & 0 & \omega_z & -\omega_y \\ \omega_y & -\omega_z & 0 & \omega_x \\ \omega_z & \omega_y & -\omega_x & 0 \end{bmatrix} \tag{6}$$

$$\omega = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}, \quad u = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}, \quad I = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}. \tag{7}$$

Note that the equations of motion in both (5a) and (5b) are nonlinear, rendering linear optimization methods inapplicable. The nonlinearities are, however, "smooth" as required for applicability of Theorem 2.

The admissible control set is characterized by

$$|u_x|,\ |u_y|,\ |u_z| \leq u_{max}\,. \tag{8}$$

The interval of optimization $[t_0, t_f]$ is divided into time knots $t_0 < t_1 < \cdots < t_K = t_f$ at which waypoint constraints are set so that $q(t_k) = q_k$ for $k = 1, \ldots, K$. No constraints are set on the angular velocity ω at the knots.

From (3), the Hamiltonian becomes

$$\begin{aligned} H = \gamma &+ u_x^2 + u_y^2 + u_z^2 - \tfrac{1}{2}\lambda_1(\omega_x q_1 + \omega_y q_2 + \omega_z q_3) \\ &+ \tfrac{1}{2}\lambda_2(\omega_x q_0 + \omega_z q_2 - \omega_x q_2) \\ &+ \tfrac{1}{2}\lambda_3(\omega_y q_0 - \omega_z q_1 + \omega_x q_3) \\ &+ \tfrac{1}{2}\lambda_4(\omega_z q_0 + \omega_y q_1 - \omega_x q_2) \\ &+ \lambda_5\left(\frac{I_{yy} - I_{zz}}{I_{xx}}\omega_y\omega_z + \frac{1}{I_{xx}}u_x\right) \\ &+ \lambda_6\left(\frac{I_{zz} - I_{xx}}{I_{yy}}\omega_x\omega_z + \frac{1}{I_{yy}}u_y\right) \\ &+ \lambda_7\left(\frac{I_{xx} - I_{yy}}{I_{zz}}\omega_x\omega_y + \frac{1}{I_{zz}}u_z\right). \end{aligned} \tag{9}$$

The adjoint equations are now formed from the Hamiltonian using Condition 1 of Theorem 2:

$$\begin{aligned} \dot{\lambda}_1 &= -\tfrac{1}{2}(\lambda_2\omega_x + \lambda_3\omega_y + \lambda_4\omega_z) \\ \dot{\lambda}_2 &= -\tfrac{1}{2}(-\lambda_1\omega_x - \lambda_3\omega_z + \lambda_4\omega_y) \\ \dot{\lambda}_3 &= -\tfrac{1}{2}(-\lambda_1\omega_y + \lambda_2\omega_z - \lambda_4\omega_x) \\ \dot{\lambda}_4 &= -\tfrac{1}{2}(-\lambda_1\omega_z - \lambda_2\omega_y - \lambda_3\omega_x) \\ \dot{\lambda}_5 &= -\tfrac{1}{2}(-\lambda_1 q_1 + \lambda_2 q_0 + \lambda_3 q_3 + \lambda_4 q_2) \\ &\quad - \frac{I_{zz} - I_{xx}}{I_{yy}}\lambda_6\omega_z - \frac{I_{xx} - I_{yy}}{I_{zz}}\lambda_7\omega_y \\ \dot{\lambda}_6 &= -\tfrac{1}{2}(-\lambda_1 q_2 - \lambda_2 q_3 + \lambda_3 q_0 + \lambda_4 q_1) \\ &\quad - \frac{I_{yy} - I_{zz}}{I_{xx}}\lambda_5\omega_z - \frac{I_{xx} - I_{yy}}{I_{zz}}\lambda_7\omega_x \\ \dot{\lambda}_7 &= -\tfrac{1}{2}(-\lambda_1 q_3 + \lambda_2 q_2 - \lambda_3 q_1 + \lambda_4 q_0) \\ &\quad - \frac{I_{yy} - I_{zz}}{I_{xx}}\lambda_5\omega_y - \frac{I_{zz} - I_{xx}}{I_{yy}}\lambda_6\omega_x \end{aligned} \tag{10}$$

The minimum condition is then applied from Condition 2 of Theorem 2, resulting in closed-form expressions for the optimal control:

$$u_x = \begin{cases} 1\,, & \lambda_5 < -2I_{xx} \\ -\lambda_5/2I_{xx}\,, & -2I_{xx} \leq \lambda_5 < 2I_{xx} \\ -1\,, & 2I_{xx} \leq \lambda_5 \end{cases} \tag{11a}$$

$$u_y = \begin{cases} 1\,, & \lambda_6 < -2I_{yy} \\ -\lambda_6/2I_{yy}\,, & -2I_{yy} \leq \lambda_6 < 2I_{yy} \\ -1\,, & 2I_{yy} \leq \lambda_6 \end{cases} \tag{11b}$$

$$u_z = \begin{cases} 1\,, & \lambda_7 < -2I_{zz} \\ -\lambda_7/2I_{zz}\,, & -2I_{zz} \leq \lambda_7 < 2I_{zz} \\ -1\,, & 2I_{zz} \leq \lambda_7 \end{cases} \tag{11c}$$

Note that these equations express the control as a function of the costate, thus eliminating the control from the set of differential equations. Note also that since the costate is continuous within each subinterval, the optimal control is also. It can be shown that the optimal control is continuous across the knots also, and this is the result of transversality.

Condition 3 of Theorem 2 gives the transversality conditions. With waypoint constraints and transversality, there are $2nK$ boundary conditions which we tabulate as follows. The state boundary conditions are

- Waypoint constraints at all knots $\left(4(K+1)\right)$, and
- Continuity of all states at interior knots $\left(7(K-1)\right)$.

The costate boundary conditions are

- Zero angular velocity costate at t_0 and t_f (6), and
- Continuity of angular velocity costate at interior knots $\left(3(K-1)\right)$.

This totals $14K$ boundary conditions which appear to balance with that required, since there are $n=7$ states. But Condition 4 of Theorem 2 imposes K additional boundary conditions: the Hamiltonian is zero at the termination of each subinterval.

Adding up the boundary conditions, we apparently have K too many. This is explained by the fact that there are K additional unknowns when the knots are free ($T_k = t_k - t_{k-1}$ are the unknowns).

Summarizing the count, we have 14 differential equations on K intervals plus K unknown time intervals T_k. There are $15K$ boundary conditions, exactly as needed, thus we have a well-defined multipoint boundary value problem.

CHALLENGE OF NUMERICAL SOLUTION

For an initial value problem, the solution to a differential equation can be directly integrated starting at the given initial condition. It is a very different situation for a multipoint boundary value problem in that it may not have a solution, and if it does, it may not be unique. Methods of numerical solution are generally classified as direct or indirect [4].

Direct methods optimize a problem by creating a grid upon which the system is discretized, and the resulting finite dimensional system is optimized. Some direct methods create adjoint equations internally [5], in analogy with optimization by the method of Lagrange multipliers.

Applying Pontryagin's Maximum Principle to infer an optimal solution is an indirect method. The resulting multipoint boundary value problem is then solved for using such methods as shooting [6], which tend to be numerically unstable, or collocation [7], which generally are stable numerically. Although the continuous nature of the differential equations is retained, a computational issue with indirect methods is the doubling of the state size, when the state equations are adjoined with costate equations. Another issue with indirect methods is that they generally require an initial guess that is sufficiently close to the optimal solution in order to converge to the proper solution.

Our first attempt at solution was to use the indirect method of solving the multipoint boundary value problem using Matlab function `bvp4c` [8]. Function `bvp4c` is a fourth-order collocation method and has the capability to solve the multipoint problem, but the time knots must be fixed. `bvp4c` accommodates unknown parameters, however, and we were successful in setting up the lengths of the time intervals, T_k, as unknown parameters, which transforms the free time problem into a fixed time problem.

At this stage of research, we are encountering difficulties in coming up with the sufficiently close initial guess required by `bvp4c`. An heuristic guess for the optimal state is easy to formulate, but an initial guess for the costate is much more difficult to glean. Generating the costate guess by random numbers works in some cases, but only when the state size and number of intervals is small. The most difficult part of numerical solution thus is coming up with a sufficiently close initial guess of the costate.

SUMMARY AND CONCLUSION

A new result that extends Pontryagin's Maximum Principle was introduced and was applied to a satellite attitude scheduling problem. The theory behind the new result is complete, and will soon be published in [2].

Difficulty was encountered in finding the optimal solution by numerical means. An indirect method was used in which the problem is set up as a well-defined multipoint boundary value problem. An initial guess is required by the software used, but our ad hoc method of constructing an initial guess was not sufficiently close to the optimal solution for this application. We have been successful in solving problems with smaller state size, but when the state size exceeds $n=4$, solution by this method becomes more of a challenge.

Our current research is focused on developing an adequate heuristic for optimal costate. One idea is to exploit the cost sensitivity interpretation of the costate, in which physical intuition might admit a close enough initial condition. Another is to use a direct method that creates the costate, for example as described in [5], and use it as the heuristic. The indirect method would then be used to verify the optimal solution.

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