Optimal Waterway and Harbor Navigation of Large Vessels

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BIOGRAPHIES

Alan Zorn is a Ph.D. candidate at Stanford University in the Aeronautics and Astronautics Engineering program. His dissertation focuses on the optimal control of nonlinear systems with multipoint constraints. Applications range from trajectory optimization to gravity-assisted space flight. Prior to his retirement into academia, he was Manager of the Navigation Systems Department at Dynamics Research Corporation where he provided navigation analysis, test, and software services for the Trident Fleet Ballistic Missile Program. He also chaired the New England Section of the Institute of Navigation. Mr. Zorn holds an M.S. in Aeronautics and Astronautics Engineering from Stanford University, an M.S.E. in Computer, Information, and Control Engineering from the University of Michigan, and a B.S. in Electrical Engineering from Lehigh University.

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ABSTRACT

A new result in optimal control provides a unified approach to optimizing the navigation of large vessels as they pass through waterways and harbors to and from their berthings. Depending on vessel size and channel characteristics, significant time and fuel can be expended to safely guide the course and speed along a specified path. Safe paths have historically been developed using well-established heuristic relationships involving channel depth, obstacle/traffic clearance, traffic volume, water current, tides, visibility, and weather. Typically a large vessel is guided through a set of waypoints to arrive at or depart from its berthing. Between waypoints, however, some flexibility in the path is permitted. The objective of this study is to utilize this flexibility to optimize the ship trajectory between waypoints to minimize time or fuel, in the absence of other vessel traffic. Ship paths are tailored to vessel characteristics such as length, draft, and displacement. The multiple-interval generalization of Pontryagin's Maximum Principle, established in a pending PhD dissertation, is proposed to find the optimal trajectory of the vessel. The generalization addresses the total navigation problem, from harbor entrance to berthing, and optimizes the path accordingly. An example based on design characteristics of a Panamax cargo ship is set up, and solution methods are explored.

INTRODUCTION

Pontryagin's Maximum Principle was developed in the 1950's in response to rapid advances made in missile technology. As the Cold War escalated, it became possible to deliver missiles long distances with reasonable accuracies. A theory was needed on how to deliver the missiles optimally. The less time it takes for a missile to arrive at its target, the more military value it has; the less fuel it expends, the lighter it can be hence increasing target range.

Lev Pontryagin (1908-1988) and his assistants solved the problem in the 1950's, providing the theoretical framework for this and other similar problems [1]. The theory prescribes a well-defined set of differential equations and boundary conditions which generally leads to a solution that both exists and is unique.

Pontryagin's Maximum Principle is a collection of necessary conditions for optimal control that best transfers a linear or nonlinear dynamical system from one state to another. The principle accommodates state and control constraints. It is a variational method that identifies local extrema, but frequently it finds the global extreme. The principle is closely related to the calculus of variations, the method of Lagrange multipliers, and the Karush–Kuhn–Tucker conditions. The main power of the principle is that it reduces an infinite-dimensional function space problem to one of finite dimensions.

Recently, a new result in the optimal control of nonlinear systems was developed by the authors [2]. The new result extends Pontryagin's Maximum Principle to apply to multiple intervals. The generalization also accommodates state constraint interdependencies and parametric optimization. From interval to interval, everything about the problem can be changed, including the differential equations, the state size, the set of admissible controls, the performance criterion, and the boundary conditions.

Some portions of the new result appear in previous works, such as interior point equality constraints [3]. For the most part, however, they have been applied by appeal to the intuition, rather than by rigorous proof. We do not dispute their validity and, in fact, our theorem legitimizes their use and extends their applicability considerably.

Our new result unifies theory across several fields and has diverse applications. It allows for periodic control as well as network optimization, as illustrated in Fig. 1.

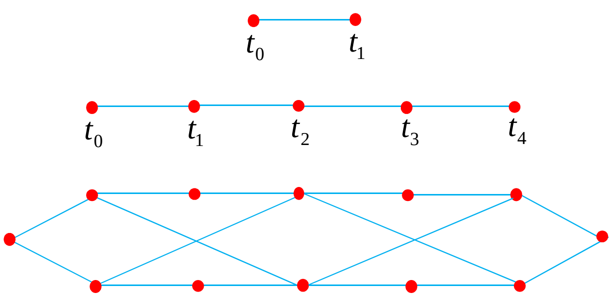

Figure 1. The top subfigure represents single-interval control. State boundary conditions are represented by the red dots and differential equations by the blue lines. The new result generalizes this to multiple intervals, pictured in the middle subfigure. The bottom subfigure represents a network that can be optimized using the new result.

Applications of the new result include:

- Optimal steering of vehicles through waypoints,
- Optimal scheduling of an imaging satellite,
- Optimal grand tour of the solar system using planetary gravity assists,
- Optimal periodic control such as transoceanic flight of an albatross or repetitive tasks in automated manufacturing,
- Generalized spline interpolation,
- Optimal systems of partial differential equations, and
- Optimal base running for inside the park home run.

In this paper, we apply the new theory to optimally control the path of large vessels as they pass waypoints in waterways and harbors. The control is optimal with respect to time or energy, or some weighted combination thereof determined by the designer. The equations of motion are nonlinear, which the theory accommodates. The goal of the paper is to report the new result in the context of ship navigation, set up the multipoint boundary value problem implied by the new result, and discuss possible ways to solve the resulting equations.

PONTRYAGIN'S MAXIMUM PRINCIPLE

Consider the system of differential equations

$$\dot{x} = f(x, u) \tag{1}$$

defined on the interval $I = [t_0, t_f]$, where $x : I \to R^n$ is the state and $u : I \to R^m$ is the control. u is an *admissible control* if it is Lebesgue measurable, bounded on I, and $u(t) \in U$ for all t. U is an arbitrary specified subset of R^m. When u is an admissible control and f is a sufficiently smooth function of x and u, (1) generally has a unique solution on interval I for any given initial condition.

Solutions x of (1) are deemed *feasible* if there exists an admissible control u that drives the state from some initial condition $x(t_0) \in M_0$ to some final condition $x(t_\mathrm{f}) \in M_\mathrm{f}$, where M_0 and M_f are given smooth manifolds in R^n. A manifold is simply a smooth surface in R^n, an example of which is the unit sphere in R^3. A manifold is typically defined by a set of k scalar algebraic equations, in which case the manifold is said to have dimension $n - k$. For example, the unit sphere in R^3 is defined by the single equation $x^2 + y^2 + z^2 = 1$, so it is a 2-dimensional manifold. The introduction of M_0 and M_f allow us to accommodate incompletely specified state in a fairly general manner. If the above conditions are satisfied, we say that (x, u) is a *feasible pair*.

Consider feasible pairs (x, u), when they exist, that minimize

$$J(x, u) = \int_{t_0}^{t_\mathrm{f}} f^0\,(x(t), u(t))dt \tag{2}$$

where f^0 is a sufficiently smooth function of x and u. In this case, (x, u) is said to be an *optimal pair*. When an optimal pair (x, u) exists, it attains the global minimum of $J(x, u)$ over all feasible pairs, but it may not be unique. Even if feasible pairs exist, there is no guarantee that any are optimal.

The Hamiltonian is defined as the scalar function

$$H(x, u, \lambda^0, \lambda) = \lambda^0\, f^0(x, u) + \lambda^\mathrm{T} f(x, u)$$

where $\lambda : I \to R^n$ is called the *costate*. In this and other similar applications, we can set the scalar $\lambda^0 = 1$ without significant loss of generalization. Doing so simplifies the Hamiltonian to

$$H(x, u, \lambda) = f^0(x, u) + \lambda^{\mathrm{T}} f(x, u) \tag{3}$$

which we use henceforth. Roughly speaking, the Hamiltonian is the infinite-dimensional analogue of adjoining a finite dimensional constrained objective function with Lagrange multipliers.

Transversality is now defined. Let M be a manifold in R^n. The tangent plane of a manifold M at a point $x \in M$ is denoted by $T_x M$. A vector $\lambda \in R^n$ satisfies the *transversality condition* with respect to M at x if $\lambda^T \mu = 0$ for all $\mu \in T_x M$. The condition is equivalently stated as $\lambda \perp T_x M$. When manifold M consists of a single point, for example, the transversality condition is vacuous. At the other extreme, when $M = R^n$, the transversality condition requires $\lambda = 0$.

A version of the original Pontryagin's Maximum Principle useful for our purposes is now stated. Some of the more esoteric technical conditions are omitted for clarity, and can be found in [1].

Theorem 1. Let $\dot{x} = f(x, u)$ be defined on the interval $I = [t_0, t_{\mathrm{f}}]$ and suppose M_0 and M_{f} are manifolds that encode initial and final conditions, respectively. If (x, u) is an optimal pair, then there exists an absolutely continuous function $\lambda : I \to R^n$ such that

1. The adjoint equations $\dot{\lambda} = -\frac{\partial H}{\partial x}$ hold a.e. on I,
2. The minimum condition
$$H\big(x(t), u(t), \lambda(t)\big) = \min_{v \in U} H\big(x(t), v, \lambda(t)\big)$$
holds for almost every $t \in I$,
3. $\lambda(t_0)$ satisfies the transversality condition with respect to M_0 at $x(t_0)$ and $\lambda(t_{\mathrm{f}})$ satisfies the transversality condition with respect to M_{f} at $x(t_0)$,
4. When the times t_0 and t_{f} are free,
$$H\big(x(t_{\mathrm{f}}), u(t_{\mathrm{f}}), \lambda(t_{\mathrm{f}})\big) = 0.$$

Furthermore, for any absolutely continuous function λ that satisfies conditions 1 and 2, the time function $H\big(x(t), u(t), \lambda(t)\big)$ is constant on I.

Application of Theorem 1 reveals that there are $2n$ first-order differential equations, n state equations and n adjoint (or costate) equations. To solve them, $2n$ boundary conditions are needed, no more and no less. When the initial and final states are specified, this creates the required $2n$ conditions. When either or both the initial or final state are not fully specified (i.e., some states are left free to be optimized), there are less than $2n$ state boundary conditions. The remaining boundary conditions "migrate" one-for-one to costate boundary conditions through transversality. The result is a two-point boundary value problem with the number of first-order differential equations precisely equal to the number of boundary conditions.

As mentioned in the Introduction, the usefulness of Pontryagin's Maximum Principle is that it reduces an infinite dimensional optimization problem to a finite dimensional problem. For well-posed problems, application of these necessary conditions typically identifies a single feasible pair (x, u) which is the only possible optimal solution. Theorem 1 does not guarantee that this solution is optimal, but it does guarantee either that it is optimal or that no optimal solution exists. Other means, usually knowledge of the application, is then used to resolve between the two possibilities.

Two examples demonstrating Pontryagin's Maximum Principle are illustrated in Figs. 2 and 3.

MULTIPLE INTERVAL EXTENSION OF PONTRYAGIN'S MAXIMUM PRINCIPLE

In [2], we generalize Pontryagin's Maximum Principle to apply to an interval $I = [t_0, t_f]$ partitioned into a grid of knots $t_0 < t_1 < \cdots < t_K = t_f$. The knots can be fixed or free. Denote the closed subintervals of the grid by $I_k = [t_{k-1}, t_k]$ for $k = 1, \ldots, K$. Constraints, such as interior point equality constraints (waypoints), can be applied at the knots. As mentioned in the Introduction, system characteristics can also differ from subinterval to subinterval. In this paper, we demonstrate the multipoint

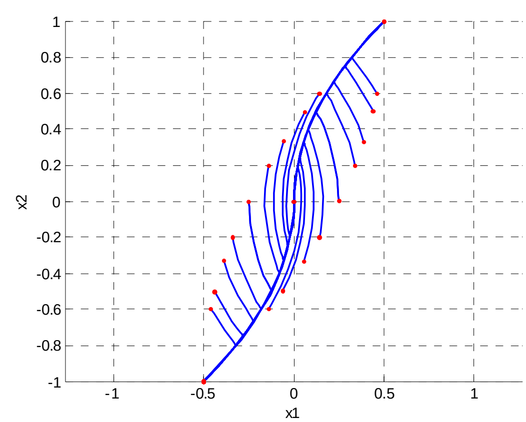

Figure 2. Time optimal control of a double integrator. Phase plane shows optimal trajectory, x_1 is position and x_2 is velocity. The red dots represent various initial conditions. The optimal control is $u_{\max}$ until reaching switching curve, then it jumps to $u_{\min}$ driving the state to the origin in minimum time. Pontryagin formalized the theory behind bang-bang control, and extended it to many other important problems.

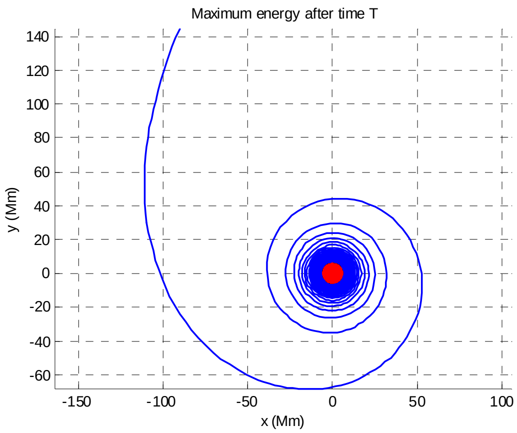

Figure 3. Optimal thrust control of spacecraft to reach escape velocity. Constant thrust ion engine powers spacecraft to maximize orbital energy after time T. Related problems that can be solved using Pontryagin's Maximum Principle are minimum time to escape velocity, minimum fuel to escape, and minimum constant thrust to escape in time T.

application without changing system characteristics across the subintervals.

Our new result affirms formation of the Hamiltonian of Theorem 1 from which both the adjoint equations and minimum condition follow. What is new is how the transversality condition is applied. We now state the multiple interval extension of Pontryagin's Maximum Principle in a form useful for the next section.

Theorem 2. Let $\dot{x} = f(x,u)$ be defined on the interval $I = [t_0, t_f]$ which is partitioned into a grid of knots $t_0 < t_1 < \cdots < t_K = t_f$. Also let $I_k = [t_{k-1}, t_k]$ for $k = 1, \ldots, K$. Suppose $\{M_k\}_0^K$ are manifolds into which boundary conditions at the knots have been encoded. If (x,u) is an optimal pair, then there exists a function $\lambda : I \to R^n$ that is absolutely continuous everywhere except possibly at the knots such that

1. The adjoint equations $\dot{\lambda} = -\frac{\partial H}{\partial x}$ hold a.e. on I,
2. The minimum condition
$$H\big(x(t), u(t), \lambda(t)\big) = \min_{v \in U} H\big(x(t), v, \lambda(t)\big)$$
holds for almost every $t \in I$,
3. The transversality conditions are satisfied:
$$\big(\lambda(t_0), -\lambda(t_1^-), \lambda(t_1^+), -\lambda(t_2^-), \ldots, \lambda(t_{K-1}^+), -\lambda(t_K)\big)$$
is orthogonal to the tangent space of M at the point
$$\big(x(t_0), x(t_1^-), x(t_1^+), x(t_2^-), \ldots, x(t_{K-1}^+), x(t_K)\big)$$
where
$$M = M_0 \times M_1 \times M_1 \times M_2 \times \cdots \times M_{K-1} \times M_K,$$
and
4. When all the knots t_k are free,
$$H\big(x(t_k^-), u(t_k^-), \lambda(t_k^-)\big) = 0 \text{ for } k = 1, \ldots, K.$$

Furthermore, for any absolutely continuous (except possibly at the knots) function λ that satisfies Conditions 1 and 2, the time function $H\big(x(t), u(t), \lambda(t)\big)$ is constant on each I_k.

Application of Theorem 2 reveals $2nK$ first-order differential equations (n state equations plus n adjoint equations, on each of the K intervals). To solve them, exactly $2nK$ boundary conditions are needed. When the initial and final states are specified on each subinterval I_K, this creates the required $2nK$ conditions. But then Theorem 2 offers nothing new, since Theorem 1 can be applied K times to obtain the optimal solution.

The power of Theorem 2 arises with incompletely specified state constraints, for example when position, but not velocity, is specified at the knots. For incompletely specified states, the theorem provides a recipe for constructing the $2nK$ boundary conditions. As before, the boundary conditions associated with unspecified states "migrate" one-for-one to costate boundary conditions through transversality. For the unspecified states, continuity of state must be enforced. It is noteworthy that Theorem 1 accomplishes this simply by "migrating" continuity of state to continuity of costate. Again, Theorem 1 prescribes exactly $2nK$ boundary conditions. The result is a multipoint boundary value problem with the number of first-order differential equations precisely equal to the number of boundary conditions.

Figure 4 shows an aircraft pylon race optimized using the new result, an example of optimal waypoint steering.

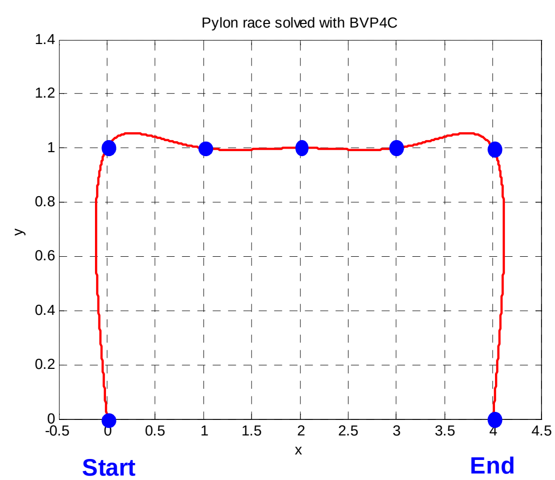

Figure 4. Aircraft races from pylon to pylon to minimize time on course. Aircraft has limited thrust, but can steer in any direction. In this multipoint optimization, there are seven waypoints, the pylons, and six subintervals ($K = 6$). Velocities at all waypoints, including start and end, are unspecified.

SHIP EQUATIONS OF MOTION

A simplified model of the motion of a ship traveling at low speeds is suggested in [4]. This model was reformulated into the following 5-state model:

$$
\begin{aligned}
&M\dot{v} + g(v - w_A) = F \\
&I\dot{\omega} + h(\omega - \omega_R) = N \\
&\dot{r}_N = v\cos\theta + w_N(r_N, r_E) \\
&\dot{r}_E = v\sin\theta + w_E(r_N, r_E) \\
&\dot{\theta} = \omega\,. \qquad (4)
\end{aligned}
$$

The variables and functions are defined in Tables 1 and 2. Values associated with a Panamax class ship are given in Table 3.

Note that the equations of motion in (4) are nonlinear, rendering linear optimization methods inapplicable. The nonlinearities are, however, "smooth" as required for applicability of Theorem 2.

A few comments are in order about simplifying assumptions made that led to the model given in (4):

- At slow speed, wave action is small. The model thus does not include roll, pitch, heave, or sway.
- Side slip motion of the ship (relative to the water) is zero. This assumption is equivalent to infinite hull resistance in the lateral (athwartship) direction. The ship speed relative to the water, v_A, is thus the longitudinal (along ship) component of speed relative to the water.
- Water current (w_N, w_E) is modeled as a function of ship position (r_N, r_E) to allow spatially-varying water current fields such as river and tidal flow. The water current field is not time varying.
- Typical thrusting hull resistance of a large ship is exemplified in Fig. 5. At slow ship speeds in harbors and waterways, thrusting hull resistance can be approximated by viscous friction $g(\Delta v) = K_v \Delta v|\Delta v|$ where $\Delta v = v - w_A$ is the longitudinal speed relative to the water. Turning hull resistance can similarly be approximated by viscous friction $h(\Delta\omega) = K_\omega \Delta\omega|\Delta\omega|$ where $\Delta\omega = \omega - \omega_R$.

It is useful to regard propeller thrust, F, and rudder torque, N, as convenient functions of control variables described as follows. Instead of controlling thrust directly, it can be controlled by a desired or commanded steady-state velocity:

$$F = g(v_C) \qquad (5)$$

It can be seen from (1) that, in the absence of water current, a constant setting of v_C in (5) eventually results in a ship speed of v_C (g must be injective as it is here). This way, posted waterway speed limits are imposed by the theory simply in requiring $|v_C| \le v_{max}$. Similarly, rudder torque can be controlled by

$$N = h(r_C v) \qquad (6)$$

Table 1
State Variables

State	Description
v	Longitudinal ship speed relative to the water (kt)
ω	Heading rate (deg/h)
r_N	North position (nm)
r_E	East position (nm)
θ	Heading (deg)

Table 2
Other Variables and Functions

Symbol	Description
M	Mass of ship (kg)
I	Moment of inertia of ship about vertical
F	Propeller thrust
N	Rudder torque
g	Hull resistance function (thrusting)
h	Hull resistance function (turning)
w_N	North water current speed (kt)
w_E	East water current speed (kt)
w_A	Longitudinal water current speed (kt) $w_A = w_N\cos\theta + w_E\sin\theta$
ω_R	Water current rotation rate (deg/h) $\omega_R = \frac{1}{2}\left(\frac{\partial w_E}{\partial x} - \frac{\partial w_N}{\partial y}\right)$

Table 3
Panamax Container Ship Characteristics

Characteristic	Value/Variable
Displacement	M = 83,000 kg
Length	L = 228 m
Beam	B = 32 m
Draft	D = 12 m
Moment of inertia about vertical	$I = M(L^2 + B^2)/12$
Max propeller thrust	F_{max} in N
Max rudder torque	N in N·m
Linear coefficient of viscous friction	K_v in N/kt^2
Angular coefficient of viscous friction	K_ω in N·m/(deg/h)^2

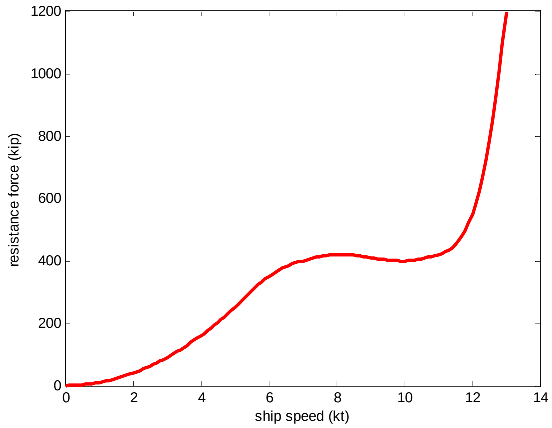

Figure 5. Total hull resistance (thrusting) as function of ship speed. At slow speeds, it can be modeled as viscous friction, proportional to the square of ship speed.

where $R_C = 1/r_C$ is the commanded radius of curvature. To better understand this, let R be the radius of curvature of a ship turn and $r = 1/R$. At any ship speed, v, the turn rate satisfies $R\omega = v$, thus $\omega = rv$ and $h(\omega) = h(rv)$ which leads to (6). There is a maximum rudder angle (let $r_{max} = 1/R_{min}$), so that it is natural to impose this limitation as $|r_C| \leq r_{max}$. This also nicely fits in with the theory.

OPTIMAL CONTROL OF SHIP TRAJECTORY

Theorem 2 is now applied to optimize the trajectory of a large ship. We choose to minimize a weighted combination of energy expended and elapsed time according to a performance criterion given by

$$J = \int_{t_0}^{t_f} \{\gamma + (C_v v_C)^2 + (C_r r_C)^2\}\, dt \qquad (7)$$

where C_v and C_r scale the two controls. The parameter $\gamma \geq 0$ weighs the criterion between energy optimal (small γ) and time optimal (large γ), and takes on a fixed value assigned by the designer.

The admissible control set is characterized by

$$|v_C| \leq v_{max} \text{ and } |r_C| \leq r_{max}\,. \qquad (8)$$

For now, we assume no water current (no flow, no rotation).

The interval of optimization $[t_0, t_f]$ is divided into time knots $t_0 < t_1 < \cdots < t_K = t_f$ at which waypoint constraints are set so that $r_N(t_k) = r_{N,k}$ and $r_E(t_k) = r_{E,k}$ for $k = 1, \ldots, K$. No constraints are set on v, ω, and θ at the knots.

From (3), the Hamiltonian can be worked out to be

$$\begin{aligned} H = \gamma + (C_v v_C)^2 + (C_r r_C)^2 + \lambda_1 \frac{g(v_C) - g(v - w_A)}{M} \\ + \lambda_2 \frac{h(r_C v) - h(\omega - \omega_R)}{I} \\ + \lambda_3 (v \cos\theta + w_N(r_N, r_E)) \\ + \lambda_4 (v \sin\theta + w_E(r_N, r_E)) + \lambda_5 \omega \end{aligned} \qquad (9)$$

The adjoint equations can be computed analytically from the Hamiltonian using Condition 1 of Theorem 2:

$$\begin{aligned} \dot{\lambda}_1 &= -\frac{\partial H}{\partial v} \\ \dot{\lambda}_2 &= -\frac{\partial H}{\partial \omega} \\ \dot{\lambda}_3 &= -\frac{\partial H}{\partial r_N} \\ \dot{\lambda}_4 &= -\frac{\partial H}{\partial r_E} \\ \dot{\lambda}_5 &= -\frac{\partial H}{\partial \theta} \end{aligned} \qquad (10)$$

The minimum condition is then applied from Condition 2 of Theorem 2, and closed-form expressions for the speed and rudder controls can be computed from

$$v_C^{\text{opt}} = \underset{|v_C| \leq v_{max}}{\text{argmin}}\, H \qquad (11a)$$

$$r_C^{\text{opt}} = \underset{|r_C| \leq r_{max}}{\text{argmin}}\, H \qquad (11b)$$

Note that when solved, these equations express the control as a function of the costate, thus eliminating the control from the set of differential equations.

Condition 3 of Theorem 2 gives the transversality conditions. With waypoint constraints and transversality, there are $2nK$ boundary conditions which we tabulate as follows. The state boundary conditions are

- Waypoint constraints at all knots $(2(K+1))$, and
- Continuity of all states at interior knots $(5(K-1))$.

The costate boundary conditions are

- Zero v, ω, and θ costates at t_0 and t_f (6), and
- Continuity of v, ω, and θ costates at interior knots $(3(K-1))$.

This totals $10K$ boundary conditions which appear to balance with that required, since there are $n = 5$ states. But Condition 4 of Theorem 2 imposes K additional boundary conditions: the Hamiltonian is zero at the termination of each subinterval.

Adding up the boundary conditions, we apparently have K too many. This is explained by the fact that there are K additional unknowns when the knots are free ($T_k = t_k - t_{k-1}$ are the unknowns).

Summarizing the count, we have 10 differential equations on K intervals plus K unknown time intervals T_k. There are $11K$ boundary conditions, exactly as needed, thus we have a well-defined multipoint boundary value problem.

CHALLENGE OF NUMERICAL SOLUTION

For an initial value problem, the solution to a differential equation can be directly integrated starting at the given initial condition. It is a very different situation for a multipoint boundary value problem in that it may not have a solution, and if it does, it may not be unique. Methods of numerical solution are generally classified as direct or indirect [5].

Direct methods optimize a problem by creating a grid upon which the system is discretized, and the resulting finite dimensional system is optimized. Some direct methods create adjoint equations internally [6], in analogy with optimization by the method of Lagrange multipliers.

Applying Pontryagin's Maximum Principle to infer an optimal solution is an indirect method. The resulting multipoint boundary value problem is then solved for using such methods as shooting [7], which tend to be numerically unstable, or collocation [8], which generally are stable numerically. Although the continuous nature of the differential equations is retained, a computational issue with indirect methods is the doubling of the state size, when the state equations are adjoined with costate equations. Another issue with indirect methods is that they generally require an initial guess that is sufficiently close to the optimal solution in order to converge to the proper solution.

Our first attempt at solution was to use the indirect method of solving the multipoint boundary value problem using Matlab function `bvp4c` [9]. Function `bvp4c` is a fourth-order collocation method and has the capability to solve the multipoint problem, but the time knots must be fixed. `bvp4c` accommodates unknown parameters, however, and we were successful in setting up the lengths of the time intervals, T_k, as unknown parameters, which transforms the free time problem into a fixed time problem.

At this stage of research, we are encountering difficulties in coming up with the sufficiently close initial guess required by `bvp4c`. An heuristic guess for the optimal state is easy to formulate, but an initial guess for the costate is much more difficult to glean. Generating the costate guess by random numbers works in some cases, but only when the state size and number of intervals is small. The most difficult part of numerical solution thus is coming up with a sufficiently close initial guess of the costate.

SUMMARY AND CONCLUSION

A new result that extends Pontryagin's Maximum Principle was introduced and was applied to the navigation of large vessels as they pass through waterways and harbors to and from their berthings. The theory behind the new result is complete, and will soon be published in [2].

Difficulty was encountered in finding optimal solutions by numerical means. An indirect method was used in which the problem is set up as a well-defined multipoint boundary value problem. An initial guess is required by the software used, but our ad hoc method of constructing an initial guess was not sufficiently close to the optimal solution for this application. We have been successful in solving problems with smaller state size, but when the state size exceeds $n = 4$, solution by this method becomes more of a challenge.

Our current research is focused on developing an adequate heuristic for optimal costate. One idea is to exploit the cost sensitivity interpretation of the costate, in which physical intuition might admit a close enough initial condition. Another is to use a direct method that creates the costate, for example as described in [5], and use it as the heuristic. The indirect method would then be used to verify the optimal solution.

ACKNOWLEDGMENT

This work was done in partial fulfillment of the requirements for the Ph.D. degree at Stanford University.

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